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Diferencialne operacije vektorske analize

Študijsko gradivo

Če ni drugače rečeno, so v nadaljevanju vsi vektorji izraženi v standardni bazi $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$.
Krajevni vektor točke $T(x, y, z)$ je $\vec{r} = (x, y, z) = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$. V nadaljevanju pomenijo

$$\alpha, \beta, \ldots \quad \text{realne konstante,}$$

$$\mathcal{U}(\vec{r}), \mathcal{V}(\vec{r}), \ldots \quad \text{skalarna polja,}$$

$$\vec{F}(\vec{r}), \vec{G}(\vec{r}), \ldots \quad \text{vektorska polja.}$$

Osnovni diferencialni operator prvega reda

Hamiltonov operator, nabla:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Gradient skalarnega polja $\mathcal{U}$:

$$\operatorname{grad} \mathcal{U} = \vec{\nabla} \mathcal{U}\,.$$

Odvod skalarnega polja $\mathcal{U}$ v smeri enotskega vektorja $\vec{e}$:

$$\frac{\partial \mathcal{U}}{\partial \vec{e}}(\vec{r}) = \lim_{\lambda \to 0} \frac{\mathcal{U}(\vec{r} + \lambda \vec{e}) - \mathcal{U}(\vec{r})}{\lambda}\,.$$

Zaključena formula:

$$\frac{\partial \mathcal{U}}{\partial \vec{e}}(\vec{r}) = \vec{e} \operatorname{grad} \mathcal{U} = \vec{e}\,(\vec{\nabla} \mathcal{U})\,.$$

Odvod skalarnega polja $\mathcal{U}$ v smeri danega vektorja $\vec{a}$:

$$\frac{\partial \mathcal{U}}{\partial \vec{a}} = \vec{a}\,\mathrm{grad}\,\mathcal{U} = \vec{a}\,(\vec{\nabla}\mathcal{U})\,.$$

Če $\vec{a}$ ni nič, lahko pišemo $\vec{a} = |\vec{a}|\vec{e}$, kjer je $\vec{e}$ enotski vektor; potem velja:

$$\frac{\partial \mathcal{U}}{\partial \vec{a}} = |\vec{a}|\vec{e}\,\mathrm{grad}\,\mathcal{U} = |\vec{a}|\frac{\partial \mathcal{U}}{\partial \vec{e}}\,.$$

Odvod danega vektorskega polja $\vec{F}$ v smeri drugega danega vektorskega polja $\vec{G}$:

Naj bo $\vec{F} = (P, Q, R)$. Potem je

$$\frac{\partial \vec{F}}{\partial \vec{G}} = \left(\frac{\partial P}{\partial \vec{G}}, \frac{\partial Q}{\partial \vec{G}}, \frac{\partial R}{\partial \vec{G}}\right)\,,$$

kjer so komponente vektorja na desni strani definirane kot odvodi komponent polja $\vec{F}$ po polju $\vec{G}$. Torej lahko izrazimo:

$$\frac{\partial \vec{F}}{\partial \vec{G}} = \left(\vec{G}\,\mathrm{grad}\,P, \vec{G}\,\mathrm{grad}\,Q, \vec{G}\,\mathrm{grad}\,R\right) = \left(\vec{G}\,(\vec{\nabla}P), \vec{G}\,(\vec{\nabla}Q), \vec{G}\,(\vec{\nabla}R)\right)\,.$$

Enak izraz bi dobili, če bi vzeli, da diferencialni operator

$$\vec{G}\,\vec{\nabla}$$

deluje zaporedno na komponente polja $\vec{F} = (P, Q, R)$. Zato lahko pišemo:

$$\frac{\partial \vec{F}}{\partial \vec{G}} = (\vec{G}\,\vec{\nabla})\,\vec{F}\,.$$

Divergenca vektorskega polja $\vec{F} = (P, Q, R)$:

$$\mathrm{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla}\,\vec{F}\,.$$

Rotor vektorskega polja $\vec{F} = (P, Q, R)$:

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla} \times \vec{F}\,.$$

Opomba: v anglosaški literaturi pišejo namesto $\operatorname{rot}\vec{F}$ drugače: $\operatorname{curl}\vec{F}$.

Osnovni diferencialni operatorji prvega reda so linearni na linearnih kombinacijah s konstantnimi koeficienti α in β in enaki nič na konstantnih poljih $\mathcal{C}$ oziroma $\vec{C}$:

$$\begin{aligned}
\operatorname{grad}(\alpha\mathcal{U} + \beta\mathcal{V}) &= \alpha \operatorname{grad}\mathcal{U} + \beta \operatorname{grad}\mathcal{V} \\
(\vec{H}\,\vec{\nabla})(\alpha\vec{F} + \beta\vec{G}) &= \alpha(\vec{H}\,\vec{\nabla})\vec{F} + \beta(\vec{H}\,\vec{\nabla})\vec{G} \\
\operatorname{div}(\alpha\vec{F} + \beta\vec{G}) &= \alpha \operatorname{div}\vec{F} + \beta \operatorname{div}\vec{G} \\
\operatorname{rot}(\alpha\vec{F} + \beta\vec{G}) &= \alpha \operatorname{rot}\vec{F} + \beta \operatorname{rot}\vec{G}
\end{aligned}$$

$$\begin{aligned}
\operatorname{grad}\mathcal{C} &= \vec{0} \\
(\vec{G}\,\vec{\nabla})\vec{C} &= \vec{0} \\
\operatorname{div}\vec{C} &= 0 \\
\operatorname{rot}\vec{C} &= \vec{0}
\end{aligned}$$

Če je $\vec{a}$ konstanten, $\vec{r}$ pa krajevni vektor, potem dobimo za osnovne operacije:

$$\operatorname{div}\vec{r} = 3\,,\ \operatorname{rot}\vec{r} = \vec{0}\,,\ \operatorname{grad}(\vec{a}\,\vec{r}) = \vec{a}\,,\ \operatorname{grad}|\vec{r}| = |\vec{r}|^{-1}\vec{r}\,,$$

$$\operatorname{div}(\vec{a} \times \vec{r}) = 0\,,\ \operatorname{rot}(\vec{a} \times \vec{r}) = 2\vec{a}\,,\ (\vec{a}\,\vec{\nabla})\vec{r} = \vec{a}\,.$$

Druge pomembne enakosti

$$\operatorname{grad}\varphi(\mathcal{U}) \;=\; \varphi'(\mathcal{U}) \operatorname{grad}\mathcal{U}$$

$$
\begin{aligned}
\operatorname{grad}(\mathcal{U}\mathcal{V}) &= \mathcal{U}\operatorname{grad}\mathcal{V} + \mathcal{V}\operatorname{grad}\mathcal{U} \\
\operatorname{grad}(\vec{F}\,\vec{G}) &= (\vec{F}\,\vec{\nabla})\vec{G} + (\vec{G}\,\vec{\nabla})\vec{F} + \vec{F}\times\operatorname{rot}\vec{G} + \vec{G}\times\operatorname{rot}\vec{F}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{div}(\mathcal{U}\vec{F}) &= \vec{F}\operatorname{grad}\mathcal{U} + \mathcal{U}\operatorname{div}\vec{F} \\
\operatorname{div}(\vec{F}\times\vec{G}) &= \vec{G}\operatorname{rot}\vec{F} - \vec{F}\operatorname{rot}\vec{G}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{rot}(\mathcal{U}\vec{F}) &= \operatorname{grad}\mathcal{U}\times\vec{F} + \mathcal{U}\operatorname{rot}\vec{F} \\
\operatorname{rot}(\vec{F}\times\vec{G}) &= (\vec{G}\,\vec{\nabla})\vec{F} - (\vec{F}\,\vec{\nabla})\vec{G} + \vec{F}\operatorname{div}\vec{G} - \vec{G}\operatorname{div}\vec{F}
\end{aligned}
$$

Osnovni diferencialni operator drugega reda

Laplaceov operator, delta:

$$\Delta = \vec{\nabla}\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.$$

Za vektorsko polje $\vec{F} = (P, Q, R)$ je smiselno definirati:

$$\Delta\vec{F} = (\Delta P, \Delta Q, \Delta R)\,.$$

Smisla nimajo:

$$\operatorname{grad}\operatorname{grad}\mathcal{U}\,,\ \operatorname{grad}\operatorname{rot}\vec{F}\,,\ \operatorname{div}\operatorname{div}\vec{F}\,,\ \operatorname{rot}\operatorname{div}\vec{F}\,.$$

Smisel pa imajo

$$\operatorname{grad}\operatorname{div}\vec{F}\,,\ \operatorname{div}\operatorname{grad}\mathcal{U}\,,\ \operatorname{div}\operatorname{rot}\vec{F}\,,\ \operatorname{rot}\operatorname{grad}\mathcal{U}\,,\ \operatorname{rot}\operatorname{rot}\vec{F}\,,$$

in sicer veljajo enakosti:

$$\begin{aligned}
\operatorname{rot}\operatorname{rot}\vec{F} &= \operatorname{grad}\operatorname{div}\vec{F} - \Delta\vec{F}\\
\operatorname{div}\operatorname{grad}\mathcal{U} &= \Delta\mathcal{U}\\
\operatorname{div}\operatorname{rot}\vec{F} &= 0\\
\operatorname{rot}\operatorname{grad}\mathcal{U} &= \vec{0}
\end{aligned}$$

CIP - Kataložni zapis o publikaciji
Narodna in univerzitetna knjižnica, Ljubljana

514.752.7(075.8)(076)

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Diferencialne operacije vektorske analize [Elektronski vir] : študijsko gradivo / Marko Razpet. - Besedilni podatki. - [Domžale : samozal.], 2006

Način dostopa (URL): `http://javor.pef.uni-lj.si/~marko/matematika/vektan.pdf`. - Opis temelji na verziji z dne 10.02.2006

ISBN 961-6589-23-7

225019648