§6. MEASURABLE SELECTIONS

1. Uniformization problem

The uniformization problem is close to the problem of finding a single-valued solution \( y = f(x) \) of an implicitly defined equation \( F(x, y) = 0 \). From the modern point of view such a problem evidently is a special case of a selection problem: find a selection of multivalued mapping \( x \mapsto \{ y \in Y \mid F(x, y) = 0 \} \). This problem was originally started without using any "selection" terms and goes back to Hadamard and Lusin.

We shall use a geometrical approach proposed by P. S. Novikov. A set \( E \subset \mathbb{R}^3 \) is said to be uniformized with respect to the \( OY \) axis, if every vertical line \( x = \text{const} \) intersects \( E \) in at most a single point. For a given planar set \( Q \subset \mathbb{R}^2 \) the uniformization problem is a problem of finding (or, proving an existence) of a subset \( E \subset Q \) such that:

1. Projection of \( E \) onto the \( OX \) axis coincides with the projection of \( Q \) onto the same line; and
2. \( E \) is uniformized with respect to axis \( OY \).

In this case \( Q \) is said to be uniformized by \( E \).

Clearly, such a general statement admits a generalization to subsets of arbitrary Cartesian product \( X \times Y \). But the uniformization problem requires the following essential additional condition. Namely, that for a given class \( \mathcal{L} \) of planar subsets one must find a "good" class \( \mathcal{M} \) of planar subsets such that every element \( Q \in \mathcal{L} \) can be uniformized by some element \( E \in \mathcal{M} \).

To formulate the first results in this area, recall that \( B(Y) \) denotes the Borel \( \sigma \)-algebra for a topological space \( Y \) and that for separable metrizable spaces, Borel subsets are also called projective sets of the class 0. The projective sets of class \( 2n + 1 \) are defined as continuous images of projective sets (in some Polish space) of the class \( 2n \) and the projective sets of class \( 2n \) are complements of projective sets of class \( 2n - 1 \), \( n \in \mathbb{N} \). The projective sets of the first class are also called analytic sets (A-sets, or Suslin sets), and the projective sets of the second class are called also CA-sets.

We shall begin by some preliminary results.

**Theorem (6.1).**

(A) \([242,382]\) Every planar Borel set \( Q \subset \mathbb{R}^2 \) can be uniformized by a planar CA-set \( E \subset Q \).

(B) \([316]\) Every planar Borel set \( Q \subset \mathbb{R}^2 \) with closed intersections with all vertical lines can be uniformized by a planar Borel set \( E \subset Q \) and the projection \( p \times Q \) onto the axis \( OX \) is a Borel set.

(C) \([315]\) (B) holds in the case when intersections of \( Q \) with all verticals are at most countable.

(D) \([14]\) (B) holds in the case when intersections of \( Q \) with all verticals are \( F_\sigma \)-sets, i.e. unions of at most countable families of closed sets.
(E) [368] (B) holds in the case when the intersections of $Q$ with all verticals are sets which admit nonempty $F_\sigma$-intersections with some open subinterval on the vertical.

(F) [315] There exists a planar Borel set $Q \subset \mathbb{R}^2$ with $p_XQ = [0,1]$ which does not admit any Borel uniformization.

(G) [211] Every planar $CA$-set (planar $A_\lambda$-set) $Q \subset \mathbb{R}^2$ can be uniformized by a planar $CA$-set $E \subset Q$.

(H) [417] Every planar $A$-set $Q \subset \mathbb{R}^2$ can be uniformized by a planar $(A_{\mu})_{\mu \in \mathbb{N}}$-set $E \subset Q$ which has the Baire property; here, $A_\mu$ is the family of all differences of $A_{\mu}$-sets.

Novikov's Theorem (6.1)(B) was the first step in this area and his proof was practically a model for all subsequent investigations. We reformulate Theorem (6.1)(B) as a selection result:

**Theorem (6.2).** Let $F : \mathbb{R} \to \mathbb{R}$ be a multivalued mapping with closed and possibly empty values. Suppose that the graph $\Gamma_F = \{(x,y) \mid y \in F(x)\}$ of the mapping $F$ is a Borel subset of $\mathbb{R}^2$. Then:

1. $\text{Dom}(F) = \{x \in \mathbb{R} \mid F(x) \neq \emptyset\}$ is a Borel subset of $\mathbb{R}$, and
2. $F$ has a Borel selection $f : \text{Dom}(F) \to \mathbb{R}$, i.e. $f(x) \in F(x)$, for all $x \in \text{Dom}(F)$.

Recall that a singlevalued mapping $f : X \to Y$ between topological spaces is called a Borel mapping if the preimage $f^{-1}(G)$ of every open set $G \subset Y$ is a Borel subset of $X$.

The Arsenin-Novikov-Scegol'kov results were generalized to $\sigma$-compact-valued mappings between Polish spaces, i.e. separable completely metrizable spaces.

**Theorem (6.3) [54].** Let $F : X \to Y$ be a multivalued mapping between Polish spaces with $\sigma$-compact and possibly empty values. Suppose that the graph $\Gamma_F$ of the mapping $F$ is a Borel subset of $X \times Y$. Then:

1. $\text{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$ is a Borel subset of $X$; and
2. $F$ has a Borel selection $f : \text{Dom}(F) \to Y$, $f(x) \in F(x)$.

Sometimes it is possible to avoid the completeness condition for the domain of the multivalued mapping. Recall that a set of $Q$ a separable metric space $X$ is said to be bianalytic if $Q$ and $X \setminus Q$ are analytic. Notation: $Q \in BA(X)$.

**Theorem (6.4) [93].** Let $X$ be a separable metrizable space, $Y$ a Polish space and $F : X \to Y$ a $\sigma$-compact-valued mapping with possibly empty values and with a bianalytic graph $\Gamma_F \subset X \times Y$. Then:

1. $\text{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$ is a bianalytic subset of $X$; and
2. $F$ has a selection $f : \text{Dom}(F) \to Y$ such that the preimage $f^{-1}(B)$ of every Borel subset $B \subset Y$ is a bianalytic subset of $X$, i.e. $f$ is $(BA(X) \otimes B(Y))$-measurable selection of $F$. 
Further results in this direction are due to Levin who replaced a separable metrizable space $X$ by a suitable measurable space $(X, \mathcal{L})$, i.e., a set $X$ with a $\sigma$-algebra $\mathcal{L}$ of subsets of $X$. Let $X$ be a nonempty set and $\mathcal{K}$ be a class of subsets of $X$ with $\emptyset \in \mathcal{K}$. We denote by $\Sigma(\mathcal{K})$ the $\sigma$-algebra generated by $\mathcal{K}$ and by $\mathcal{A}\mathcal{K}$ the class of $\mathcal{K}$-analytic subsets of $X$, i.e., the subsets representable as the result of the $\mathcal{A}$-operation on elements of $\mathcal{K}$:

$$A \in \mathcal{A}\mathcal{K} \iff A = \bigcup_{(n_k) \in \mathbb{N}^\infty} \bigcap_{k \in \mathbb{N}} B(n_1, \ldots, n_k)$$

where $B(n_1, \ldots, n_k) \in \mathcal{K}$, and $(n_k)$ is a sequence of natural numbers $n_k \in \mathbb{N}$. $\mathcal{A}$-operation was introduced by Aleksandrov in [6]. It is known [215] that analytic subsets of a separable metrizable space $M$ admit an equivalent definition as results of $\mathcal{A}$-operation on certain Borel sets. A set $B \subset X$ is called $\mathcal{K}$-bianaalytic if $B \in \mathcal{A}\mathcal{K}$ and $X \setminus B \in \mathcal{A}\mathcal{K}$; the class of all $\mathcal{K}$-bianaalytic sets is denoted by $\mathcal{B}\mathcal{A}(\mathcal{K})$. For measurable spaces $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$ the smallest $\sigma$-algebra in $X_1 \times X_2$ containing all Cartesian products $L_1 \times L_2$ with $L_1 \in L_1$, $L_2 \in L_2$, is denoted $L_1 \otimes L_2$. Finally, the Baire $\sigma$-algebra $\mathcal{B}_0(X)$ in a topological space $X$ is defined as the $\sigma$-algebra generated by the sets $f^{-1}(0)$, where $f$ is a continuous real-valued function on $X$.

**Theorem (6.5) [224,226].** Let $F : X \to Y$ be a $\sigma$-quasicompact (quasicompactness means compactness without the Hausdorff separation property) valued mapping with possibly empty values and with the graph $\Gamma_F \in \mathcal{B}\mathcal{A}(\mathcal{L} \otimes \mathcal{B}_0(Y))$, where $(X, \mathcal{L})$ is a measurable space and $Y$ is a topological space that is the image of a Cartesian product of a family of Polish spaces under some proper (i.e., preimage of quasicompacta are quasicompacta) mapping. Then:

1. $\text{Dom}(F) \in \mathcal{B}\mathcal{A}(\mathcal{L})$; and
2. $F$ has a $(\mathcal{B}\mathcal{A}(\mathcal{L}) - \mathcal{B}(Y))$-measurable selection $f$, i.e., the preimage $f^{-1}(B)$ of any Baire subset $B \subset Y$ belongs to the $\sigma$-algebra $\mathcal{B}\mathcal{A}(\mathcal{L})$.

The proof of the following theorem uses the Continuum hypothesis (CH).

**Theorem (6.6) (CH) [226].** Let $F : X \to Y$ be a $\sigma$-compact-valued mapping between compacta with possibly empty values and with the graph $\Gamma_F$ a Baire subset of $X \times Y$. Then:

1. $\text{Dom}(F)$ is a Baire subset of $X$; and
2. $F$ has a Baire (i.e., $(\mathcal{B}_0(X) - \mathcal{B}_0(Y))$-measurable) selection $f$.

If $Y$ is a dyadic compactum (i.e., $Y$ is a continuous image of some $\{0, 1\}^\omega$), then (CH) can be avoided. See also results of Evstigneev [121] in connection with the role of (CH) in measurable selection theorems for nonmetrizable compacta $Y$.

All theorems above present the two statements: one about properties of a projection of a subset of $X \times Y$ and the other, about a selection from the image of projection into the given subset of $X \times Y$. Sion [383] obtained some results in the second direction, but without metrizability restriction.
We say that a topological space \( Y \) satisfies condition (SI) if and only if \( Y \) is completely regular, has a base of cardinality at most first uncountable cardinal and every family of open subsets of \( Y \) has a countable subfamily with the same union of elements.

**Theorem (6.7).** Suppose that \( X \) is a topological space, \( Y \) satisfies condition (SI), \( Q \) is compact in \( X \times Y \), and \( P_X : X \times Y \to X \) is a projection. Then there exists a selection \( f : P_X Q \to X \times Y \) of the multivalued mapping \( P_X^{-1} : P_X Q \to X \times Y \) such that for every open set \( U \subseteq X \times Y \), the preimage \( f^{-1}(U) \subseteq X \) is an element of the \( \sigma \)-algebra, generated by all compact subsets of \( X \).

**Theorem (6.8).** Suppose that \( X \) is a Hausdorff space, \( Y \) satisfies condition (SI) and \( Q \) is analytic in \( X \times Y \). Then there exists a selection \( f \) of \( P_X^{-1} \) such that for every open set \( V \subseteq X \times Y \), the preimage \( f^{-1}(V) \subseteq X \) is an element of the \( \sigma \)-algebra, generated by all analytic subsets of \( X \).

In the last theorem, the expression “\( A \) is analytic subset of a topological Hausdorff space \( X \)” means that for some Hausdorff space \( Z \) and for some \( B \subseteq Z \), which is an element of \( K_\sigma(Z) \), there exists a continuous mapping from \( B \) onto \( A \); \( B \in K_\sigma(Z) \iff B = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} C_{ij} \), \( C_{ij} \) are compacta in \( Z \).

As a generalization of the Sion’s results it was proved in [139] that (under the continuum hypothesis CH) every upper semicontinuous compact-valued mapping from the space of irrationals to a compact (not necessarily metric) space admits a selection, which is measurable in the sense that preimages of Baire measurable sets are universally measurable, i.e. are measurable with respect to each \( \sigma \)-finite Radon measure.

2. **Measurable multivalued mappings**

Let \( (X, \mathcal{L}) \) be a measurable space, \( Y \) a separable metric space, and \( F : X \to Y \) a closed-valued mapping with possibly empty values. Consider the following properties of the mapping \( F \). As usual, \( F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\} \).

(I) \( F^{-1}(B) \in \mathcal{L} \), for every Borel set \( B \subseteq Y \);

(II) \( F^{-1}(A) \in \mathcal{L} \), for every closed set \( A \subseteq Y \);

(III) \( F^{-1}(G) \in \mathcal{L} \), for every open set \( G \subseteq Y \);

(IV) \( \text{Dom}(F) \in \mathcal{L} \) and all distance functions \( x \mapsto \text{dist}(y, F(x)) \) are measurable real-valued functions on \( \text{Dom}(F) \), for every \( y \in Y \);

(V) \( \text{Dom}(F) \in \mathcal{L} \) and there exists a sequence \( \{f_n\} \) of measurable mappings \( f_n : \text{Dom}(F) \to Y \) such that \( F(x) = \text{Cl}\{f_n(x) \mid n \in \mathbb{N}\} \), for all \( x \in \text{Dom}(F) \); and

(VI) The graph \( \Gamma_F \) is an \( (\mathcal{L} \times B(Y))\)-measurable subset of \( X \times Y \) where \( B(Y) \) is the Borel \( \sigma \)-algebra, generated by open subsets of \( Y \).
Recall that a singlevalued mapping $f$ into a topological space is said to be measurable if the preimages of open sets are measurable subsets of the domain of $f$. These properties were collected by Castaing in [59] for the case when $\text{Dom}(F) = X$ and $Y$ is complete. See also [60,180,181].

**Theorem (6.9).** If on a $\sigma$-algebra $\mathcal{L}$ of subsets of $X$ there exists a complete $\sigma$-finite measure and if $Y$ is a Polish space then all conditions (I)-(VI) above are equivalent.

A non-negative $\sigma$-additive measure $\mu : \mathcal{L} \to [0, \infty]$ is said to be complete if every subset of a set with a zero measure has also the zero measure. The $\sigma$-finiteness of the measure $\mu$ means that $X = \bigcup_{n=1}^{\infty} X_n$ with finite $\mu(X_n)$, $n \in \mathbb{N}$. The equivalence (I) $\iff$ (VI) was proved by Debreu [92].

In general, we have only the following theorem:

**Theorem (6.10).** Under the above notations the following implications hold:

(I) $\implies$ (II) $\implies$ (III) $\iff$ (IV) $\implies$ (VI).

If $Y$ is a Polish space then

(I) $\implies$ (II) $\implies$ (III) $\iff$ (IV) $\iff$ (V) $\implies$ (VI).

Sometimes a version of completeness of $\sigma$-algebra $\mathcal{L}$ can be formulated in Theorem (6.9) without the measure.

**Theorem (6.11) [227].**

1. Let $\mathcal{L} = \mathcal{A}\mathcal{C}$ and $Y$ be a metrizable analytic space. Then the properties (I)-(VI) are equivalent.

2. Let $\mathcal{L} = \mathcal{B}\mathcal{A}(\mathcal{L})$ and $Y$ be a metrizable $\sigma$-compact. Then (III) $\iff$ (V) $\iff$ (VI).

For our purpose the property (V) is of the maximal interest. It states that a measurable multivalued mapping $F$ admits a countable "dense" family of measurable selection. Such a family is often called the Castaing representation of $F$. As a special case of the implication (III) $\implies$ (V) for the case of a Polish space $Y$ we formally obtain the Kuratowski-Ryll-Nardzewski selection theorem [216]. In fact, the situation is reversed, i.e. the implication (III) $\implies$ (V) is a corollary of such a selection theorem. Observe that for the case of so-called standard measurable space such a selection theorem was in fact, proved by Rohlin [360]. Note also, that the existence of a Castaing representation in the case $X = Y = \mathbb{R}$ is a direct corollary of the Novikov theorem (6.1)(B). The similar result was also proved by Neumann [311].

So, we give a proof of this selection theorem (called the Kuratowski-Neumann-Novikov-Rohlin-Ryll-Nardzewski-Yankov theorem) for measurable multivalued mappings.

**Theorem (6.12).** Let $(X, \mathcal{L})$ be a measurable space, $Y$ a Polish space and $F : X \to Y$ a closed valued mapping with possibly empty valued and
with $F^{-1}(G) \in \mathcal{L}$, for every open $G \subseteq Y$. Then there exists a measurable single-valued mapping $f : \text{Dom}(F) \to Y$ such that $f(x) \in F(x)$, for all $x \in \text{Dom}(F)$.

**Proof.** Without loss of generality, we can assume that $\text{Dom}(F) = X$, because $\text{Dom}(F) = F^{-1}(Y) \in \mathcal{L}$. We also assume that $Y$ is completely metrizable by some metric $d$ bounded by 1.

**I. Construction**

Let:

(1) $\{y_1, y_2, \ldots, y_k, \ldots\}$ be a dense sequence of distinct points in $Y$, with a fixed denumeration order;

(2) $\sum_{k=1}^{\infty} \epsilon_k$ be a convergent series with $0 < \epsilon_{k+1} < \epsilon_k$ and $\epsilon_1 < 1$;

(3) $f_i(x) = y_k$ for all $x \in X$;

(4) For some $n \in \mathbb{N}$, there exist measurable mappings $f_1, f_2, \ldots, f_n$ from $X$ into $Y$ such that for all $x \in X$:

\[
(\star_n) \text{ dist}(f_i(x), F(x)) < \epsilon_i; \; i = 1, 2, \ldots, n;
\]

\[
(\star\star_n) \text{ dist}(f_j(x), f_{j+1}(x)) < \epsilon_j; \; j = 1, 2, \ldots, n - 1;
\]

(5) For every $x \in X$, the value $f_{n+1}(x)$ is defined as the first element of the sequence $\{y_k\}$ which belongs to the intersection

\[D(F(x), \epsilon_{n+1}) \cap D(f_n(x), \epsilon_n).
\]

We claim that then:

(a) $f_{n+1} : X \to Y$ is well-defined;

(b) $f_{n+1}$ is a measurable mapping;

(c) For $f_{n+1}$ the following inequalities hold:

\[
(\star\star_{n+1}) \text{ dist}(f_i(x), F(x)) < \epsilon_i; \; i = 1, 2, \ldots, n, n + 1
\]

\[
(\star\star\star_{n+1}) \text{ dist}(f_j(x), f_{j+1}(x)) < \epsilon_j; \; j = 1, 2, \ldots, n;
\]

(d) For every $x \in X$, there exists $\lim_{n \to \infty} f_n(x) = f(x)$; and

(e) $f : X \to Y$ is a desired selection of $S$.

**II. Verification**

(a) Follows because $F(x) \cap D(f_n(x), \epsilon_n) \neq \emptyset$, see $(\star_n)$.

(b) $f_{n+1}(X) \subseteq \{y_k\}$. So, it suffices to check only that for every $y_k$ the “level” sets $f_{n+1}^{-1}(y_k) = \{x \in X \mid f_{n+1}(x) = y_k\}$ are $\mathcal{L}$-measurable. But $f_{n+1}^{-1}(y_k) = C_{kn} \setminus (\bigcup C_{mn} \mid m < k)$, where $C_{kn} = F^{-1}(D(y_k, \epsilon_{n+1})) \cap f_{n+1}^{-1}(D(y_k, \epsilon_n)) \in \mathcal{L}$.

(c) Follows from (5).

(d) Is due to the completeness of $Y$, inequalities $(\star\star_n)$ and (2).

(e) $f$ is measurable as a pointwise limit of measurable mappings and $f(x) \in F(x)$ because of the closedness of $F(x) \subseteq Y$.

Theorem (6.12) is thus proved. ■
We give a version of the Yankov’s theorem (see Theorem (6.1)(E)) in order to emphasize that sometimes the condition of closedness of values $F(x)$, $x \in X$, can be omitted.

**Theorem (6.13).** Let $F : X \to Y$ be a mapping between Polish spaces such that the graph $\Gamma_F$ is a Suslin subset (or $\mathcal{A}$-subset) of $X \times Y$. Then $F$ admits a measurable selection.

A simultaneous generalization of the Yankov’s theorem and von Neumann selection theorem [311] can be found in [60]. We say that a Hausdorff topological space $X$ is a Suslin space if it is a continuous image of Polish space and we denote by $S(X)$ the $\sigma$-algebra, generated by Suslin subsets of $X$.

**Theorem (6.14).** Let $F : X \to Y$ be a mapping with nonempty values from a Suslin space $X$ into a topological space $X$ such that the graph $\Gamma_F$ is a Suslin space. Then there exists a sequence $\{f_n\}$ of singlevalued $(S(X) - B(Y))$-measurable selections of $F$ such that $\{f_n(x)\}$ is dense in $F(x)$, for all $x \in X$. Moreover, every $f_n$ is the limit of a sequence of singlevalued $S(X)$-measurable mappings, assuming a finite number of values.

Under the additional assumption that $\mu$ is a regular measure, i.e. $\mu(B) = \sup\{\mu(K) \mid K$ is subcompactum of $B\}$ the selections $f_n$ in Theorem (6.14) have the Lusin $C$-property.

An analogue of Theorem (6.14) for the case when $(X, \mathcal{L})$ is a measurable space, $Y$ is a Suslin space and $\Gamma_F$ can be obtained from elements of $\mathcal{L} \otimes B(Y)$ using $\mathcal{A}$-operation, was proved in [222,225].

**Definition (6.15).** A multivalued mapping $F : X \to Y$ with arbitrary values from a measurable space $(X, \mathcal{L})$ into a separable metric space, is said to be measurable (resp. weakly measurable) if $F$ has the property (II) (resp. property (III)) above.

Note that in the literature there is some disagreement concerning the use of terms “measurable”, “weak measurable”, and “strong measurable” for multivalued mappings.

A compact-valued mapping $F : X \to Y$ with $\text{Dom}(F) = X$ can be considered as a singlevalued mapping from $X$ into the set $\exp(Y)$ of all subcompacta of $Y$, endowed with the Hausdorff distance topology.

**Theorem (6.16) [60].** For a compact-valued mapping $F : X \to Y$ with $\text{Dom}(F) = X$ from a measurable space $(X, \mathcal{L})$ into a separable metric space $Y$ the following assertions are equivalent:

1. $F$ is measurable;
2. $F$ is weakly measurable; and
3. $F : X \to \exp Y$ is a measurable singlevalued mapping.
For a metric space \( X \) and bounded subset \( A \subset X \) we define the Kuratowski index as follows:
\[
\alpha(A) = \inf \{ \varepsilon | A = \bigcup_{i=1}^{n} A_i, \text{diam} A_i \leq \varepsilon, n \in \mathbb{N} \}.
\]

A “compact” version of the Castaing representation was proposed in [63].

**Theorem (6.17).** Let \((X, \mathcal{L}, \mu)\) be a \( \sigma \)-finite measure space, \( 1 \leq p \leq \infty \) and \( F : X \to \mathbb{R}^m \) a measurable nonempty and closed-valued mapping with \( \|F(x)\| \leq \ell(x) \), for all \( x \in X \) and for some \( \ell \in L_p(X, \mathbb{R}) \). Then there exists a Castaing representation \( \{f_n\}_n \) for \( F \) such that all \( f_n \) are elements of the Banach space \( L_p(X, \mathbb{R}^m) \) and the Kuratowski index \( \alpha(\{f_n\}) \) is equal to zero in \( L_p(X, \mathbb{R}^m) \).

Finally, we state the Ioffe representation theorem which, roughly speaking, states that a measurable multivalued mapping can be factorized through two parametric mappings of a Carathéodory type.

**Theorem (6.18) [186].** Let \( Y \) be a Polish (resp. compact metrizable) space, \((X, \mathcal{L})\) a measurable space and \( F : X \to Y \) a closed-valued measurable mapping with possibly empty values. Then there exists a Polish (resp. compact metrizable) space \( Z \) and a singlevalued mapping \( f : X \times Z \to Y \) such that:

1. \( f \) is continuous in \( z \) and \( \mathcal{L} \)-measurable in \( x \); and
2. For all \( x \in \text{Dom} F \), \( F(x) \) is equal to the image \( f(x, Z) \) of \( Z \).

By taking a dense countable set in \( Z \), one gets a dense countable family of measurable selections of \( F \), i.e. the Castaing representation of \( F \).

### 3. Measurable selections of semicontinuous mappings

There exists an obvious similarity between the proofs of selection theorems in the continuous and in the measurable case. More precisely, in Michael's selection theorems as well as in Kuratowski-Ryll-Nardzewski selection theorem (see Theorem (6.12)) the resulting selection is constructed as the uniform limit of a sequence of \( \varepsilon_n \)-selections of a given multivalued mapping. A natural problem is to find a simultaneous proof of both selection theorems. Such an idea was realized by Mägerl in [245]. To begin, note that the family \( T \) of all open subsets of a topological space and the family \( \mathcal{L} \) of all measurable subsets of a measure space have the following common stability (with respect to the set operations) property: \( T \) and \( \mathcal{L} \) are closed under operations of finite intersections and countable unions.

**Definition (6.19).** If \( X \) is a set and \( \mathcal{P} \) is a family of subsets of \( X \), then \( \mathcal{P} \) is called a **paving** and the pair \((X, \mathcal{P})\) is said to be **paved** if \( X \in \mathcal{P} \), \( \emptyset \in \mathcal{P} \) and \( \mathcal{P} \) is closed under finite intersections and countable unions.
**Definition (6.20).** If \((X, \mathcal{P})\) is a paved space and \(Y\) is a topological space, then a multivalued (singlevalued, as a special case) mapping \(F : X \to Y\) is said to be \(\mathcal{P}\)-measurable if \(F^{-1}(G) \in \mathcal{P}\), for every open \(G \subset Y\).

**Definition (6.21).** Let \(k\) be a cardinal number and \(n \in \mathbb{N} \cup \{\infty\}\). A paved space \((X, \mathcal{P})\) is called \((k, n)\)-paracompact if every covering \(U \subset \mathcal{P}\) of \(X\) with cardinality less than \(k\), has a refinement \(V \subset \mathcal{P}\) such that:

1. \(\dim N(V) \leq n\); and
2. There exists a \(\mathcal{P}\)-measurable mapping \(\varphi : X \to N(V)\) with \(\varphi^{-1}(St(e_V)) \subset V\), for all \(V \in \mathcal{P}\).

Here \(N(V)\) is the geometric nerve of the covering \(V\) endowed with the Whitehead topology, \(e_V\) is the vertex of \(N(V)\) which corresponds to \(V \in \mathcal{P}\) and \(St(e_V)\) is the star of the vertex \(e_V\) in the simplicial complex \(N(V)\). Let us define an abstract version of the convex hull operator.

**Definition (6.22).** Let \(H\) be a mapping which assigns to every subset \(A\) of \(Y\) a subset, \(H(A)\) of \(Y\). Then \(H\) is called a hull-operator if \(H(\{y\}) = \{y\}, y \in Y; H(A) = H(H(A)); A \subset H(A)\) and \(A \subset B\) implies that \(H(A) \subset H(B)\). A hull-operator on a topological space \(Y\) is called \(n\)-convex if for every at most \(n\)-dimensional simplicial complex \(S\) and for every mapping \(\rho\) of vertices of \(S\) into \(Y\), there exists a continuous mapping \(\tau : S \to Y\) such that \(\tau(\Delta) \subset H(\rho(V(\Delta)))\), for all simplices \(\Delta \in S\); (here \(V(\Delta)\) is the set of all vertices of \(\Delta\)).

**Definition (6.23).** For a set \(Y\) endowed with a hull operator \(H\) a pseudometric \(d\) on \(Y\) is called \(H\)-convex if for \(\varepsilon > 0\), the equality \(A = H(A)\) implies equality \(H(D(A, \varepsilon)) = D(A, \varepsilon)\). For a uniform space \(Y\) a hull operator \(H\) is called compatible with the given uniform structure if the uniformity of \(Y\) is generated by a family of \(H\)-convex pseudometrics.

**Theorem (6.24) [245].** Let \((X, \mathcal{P})\) be a \((k, n)\)-paracompact paved space, \(Y\) a \(k\)-bounded complete metric space and \(H\) an \(n\)-convex, compatible hull-operator in \(Y\). Then every \(\mathcal{P}\)-measurable mapping \(F : X \to Y\) such that \(F(x) = Cl F(x) = H(F(x)), x \in X\), admits a singlevalued \(\mathcal{P}\)-measurable selection.

In this theorem, \(k\)-boundedness of a metric space \(Y\) means the existence of \(\varepsilon\)-nets of cardinality less that \(k\), for any \(\varepsilon > 0\).

As special cases of Theorem (6.24), one can obtain Zero-dimensional selection theorem, the Convex-valued selection theorems for normal and paracompact domains, the Kuratowski-Ryll-Nardzewski theorem and some others.

The rest of this section is devoted to the “mixed” type selection theorems, which, roughly speaking, yield for semicontinuous mappings an existence of (as a rule, non-continuous but descriptive “well”) selections. A fundamental result is due to Coban [76,77].
Theorem (6.25). Let $F : X \to Y$ be a continuous closed-valued mapping from a topological space $X$ into a completely metrizable space $Y$. Then there exists a selection $f$ of $F$ such that $f^{-1}(G)$ is an $F_\sigma$-subset of $X$, whenever $G$ is open in $Y$.

Note, that $Y$ is not necessarily separable and observe that in fact the proof consists of finding a suitable selection for the hyperspace of all nonempty closed subset of $Y$ (see also §5.4).

A well-known Hausdorff theorem states that an open continuous image of a Polish space is a complete space. Hausdorff asked a question whether an open continuous image of a Borel set of class $\alpha$ is a Borel set of the same class. Generally, the answer is negative, as it was demonstrated by Keldys [197].

Theorem (6.26). Let $f : X \to Y$ be an open mapping of a metric space $X$ onto a metric space $Y$ and let preimages $f^{-1}(y)$ be complete subsets of $X$. Suppose that $X$ is a Borel set of class $\alpha \geq 2$. Then $Y$ is a Borel set of class $\leq \alpha + 1$, provided $\alpha < w_0$, and of class $\leq \alpha$ otherwise.

Theorem (6.27). Let $F : X \to Y$ be a closed-valued mapping from a perfectly normal space $X$ into a completely metrizable space $Y$ and let $p_X : \Gamma_F \to X$ be a closed mapping where $p_X$ is the natural projection of the graph $\Gamma_F$ onto $X$. Then $F$ has a singlevalued selection $f$ such that $f^{-1}(G)$ is a $F_\sigma$-subset of $X$, whenever $G$ is open in $Y$.

The next Čoban’s theorem is in some sense reminiscent of Yankov’s theorem. Let $\mathcal{F}_\rho(X)$ be the family of all differences of closed subsets of $X$ and $\mathcal{F}_{\rho_0}(X)$ a countable union of elements of $\mathcal{F}_\rho(X)$. For perfectly normal spaces $X$ we have that $\mathcal{F}_{\rho_0}(X) = \mathcal{F}_\sigma(X)$.

Theorem (6.28). Let $F : X \to Y$ be a compact-valued lower semicontinuous mapping into a metric space $Y$. Then there exists a selection of $F$ such that $f^{-1}(G) \in \mathcal{F}_{\rho_0}(X)$, whenever $G$ is open in $Y$.

Theorem (6.29). Let $F : X \to Y$ be a closed-valued lower semicontinuous mapping from a paracompact space $X$ into a completely metrizable space $Y$. Then there exists a selection $f$ of $F$ such that $f^{-1}(G)$ is an $F_\sigma$-subset of $X$, whenever $G$ is open in $Y$.

A part of Čoban’s results was generalized by Kolesnikov to the nonmetrizable ranges $Y$, more precisely, to spaces with a $G_\delta$-diagonal and GO-spaces (see [208,209]). We finish this section by a list of some further results in this direction.

Theorem (6.30) [168]. Let $F : X \to Y$ be an upper semicontinuous completely valued mapping between metric spaces. Then $F$ has a Borel class 1 selection.

Theorem (6.31) [191]. Let $F : X \to Y$ be an upper semicontinuous mapping between metric spaces. Then $F$ has a Borel class 2 selection.
Theorem (6.32) [171]. Let $F : X \rightarrow Y$ be an upper semicontinuous mapping between metric spaces and let $Y$ be an absolute retract. Then $F$ has a Baire class 1 selection.

Theorem (6.33) [192]. Let $F : X \rightarrow Y$ be an upper semicontinuous mapping from a metric space $X$ into a Banach space $Y$ endowed by the weak topology and let $F$ take values in fixed weak subcompacta of $Y$. Then $F$ has a norm Borel selection.

Theorem (6.34) [171]. Let $F : X \rightarrow Y$ be an upper semicontinuous mapping from a metric hereditary Baire space $X$ into a Banach space $Y$ endowed by the weak topology and let all values $F(x)$ be weak compacta in $Y$. Then $F$ has a norm Baire class 1 selection.

Theorem (6.35) [387]. Let $F : X \rightarrow Y$ be as in Theorem (6.33) (without completeness of $Y$). Then there exists a sequence $\{f_n\}$ of norm continuous mappings $f_n : X \rightarrow Y$ converging pointwisely in the norm to a selection $f$ (Borel class 2) of $F$.

In Theorems (6.30)–(6.35) the term Borel class 1 (resp. class 2) mapping $f$ means that $f^{-1}(G)$ is a $F_\sigma$-set (resp. $f^{-1}(G)$ is a $G_\delta$-set), whenever $G$ is open. A mapping $f$ is said to be of a Baire class 1 if it is pointwise limit of continuous mappings. See also [190] for more details.

4. Carathéodory conditions. Solutions of differential inclusions

It is well-known that a differential equation $x' = f(t, x), x(t_0) = x_0$ with a continuous right side is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) \, d\tau.$$

For a discontinuous right-hand sides $f$ one can consider the Lebesgue integral instead of the Riemann integral and obtain a solution in the Carathéodory sense.

Definition (6.36). Let $G$ be an open connected subset of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. A singlevalued function $f : G \rightarrow \mathbb{R}$ is called a Carathéodory function if:

(a) For almost all $t \in \mathbb{R}$, the function $f(t, \cdot)$ is continuous, over $x \in \mathbb{R}^n$, where $(t, x) \in G$;
(b) For every $x$ the function $f(\cdot, x)$ is measurable over $t \in \mathbb{R}^n$, where $(t, x) \in G$; and
(c) $\|f(t, x)\| \leq \ell(t)$ for some summable function $\ell(t)$ (at each finite segment, if $G$ is unbounded along the variable $t$).
Observe, that sometimes Carathéodory conditions are stated as (a) and (b) only, while (c) is often called the \textit{integral boundedness} condition.

**Theorem (6.37)** [86]. Let \([t_0, t_0 + a] \times \text{Cl}(D(x_0, b)) \subset G \subset \mathbb{R} \times \mathbb{R}^n\) and \(f : G \to \mathbb{R}^m\) a Carathéodory function. Then for some \(d > 0\), there exists on the segment \([t_0, t_0 + d]\) an absolutely continuous function \(x(t)\) such that \(x(t_0) = x_0\) and \(x'(t) = f(t, x(t))\), for almost all \(t \in [t_0, t_0 + d]\). Moreover, one can assume that \(0 < d \leq a\) and \(\int_{t_0}^{t_0 + d} f(t, x(t)) \, dt \leq b\).

We see from \(|x(t) - x_0| = \|\int_{t_0}^{t} f(t, x(t)) \, dt\| \leq \int_{t_0}^{t_0 + d} f(t, x(t)) \, dt \leq b\) that the graph of the solution \(x(\cdot)\) from Theorem (6.37) is a subset of the rectangle \([t_0, t_0 + d] \times \text{Cl}(D(x_0, b))\).

For multivalued right-hand sides, i.e. for \textit{differential inclusions} \(x' \in F(x, t)\), there exists a series of different existence theorems. To formulate one of the earliest variants, we remark that an upper semicontinuous compact-valued mapping \(F : K \to \mathbb{R}^n\) with a metric compact domain \(K\) is \textit{bounded}, i.e. \(\sup\{\|F(k)\| : k \in K\} < \infty\).

**Theorem (6.38)** [250,420]. Let \(\Pi = [t_0, t_0 + a] \times \text{Cl}(D(x_0, b)) \subset \mathbb{R} \times \mathbb{R}^n\) and let \(F : \Pi \to \mathbb{R}^m\) be an upper semicontinuous compact- and convex-valued mapping. Then for some \(d > 0\), on the segment \([t_0, t_0 + d]\), there exists an absolutely continuous function \(x(t)\) such that \(x(t_0) = x_0\) and \(x'(t) \in F(t, x(t))\), for almost all \(t \in [t_0, t_0 + d]\). Moreover, one can assume that \(d = -\min\{\alpha : b/\sup\{\|F(t, x)\| : (t, x) \in \Pi\}\}\).

Certain versions were proposed in [91].

**Theorem (6.39)**. Let \(\Pi = [t_0, t_0 + a] \times \text{Cl}(D(x_0, b)) \subset \mathbb{R} \times \mathbb{R}^n\) and let \(F : \Pi \to \mathbb{R}^n\) be upper semicontinuous over \(x \in \text{Cl}(D(x_0, b))\), for almost all \(t \in [t_0, t_0 + a]\), let values of \(F\) be convex and closed and let \(F\) admit a single valued integrally bounded selection \(f : \Pi \to \mathbb{R}^n\) which is measurable with respect to \(t \in [t_0, t_0 + a]\), for every \(x \in \text{Cl}(D(x_0, b))\). Then the problem \(x' \in F(t, x), x(t_0) = x_0\) admits a solution (in the Carathéodory sense) over the segment \([t_0, t_0 + d]\), where \(d > 0\) is defined as in Theorem (6.37).

For a nonconvex-valued right-hand sides \(F(t, x)\) the upper semicontinuity hypothesis is insufficient. Its strengthening to continuity is, sometimes, sufficient.

**Theorem (6.40)** [194]. Let \(\Pi = [t_0, t_0 + a] \times \text{Cl}(D(x_0, b)) \subset \mathbb{R} \times \mathbb{R}^n\) and let \(F : \Pi \to \mathbb{R}^n\) be a closed-valued, integrally bounded (by a summable function \(\ell : [t_0, t_0 + a] \to \mathbb{R}\)) mapping which is continuous with respect to \(x\) and measurable with respect to \(t\). Then the problem \(x' \in F(t, x), x(t_0) = x_0\) admits a Carathéodory solution over the segment \([t_0, t_0 + d]\), where \(d > 0\) is defined as in Theorem (6.37).

It was shown in [319] that the continuity condition in the theorem can be weakened at the points \((t, x)\) with convex \(F(t, x)\) to the upper semicontinuity with respect to \(x\).
For the proofs of these theorems, see also [136]. We describe the sketch of a proof with an attention to selections. First, we need the notion of the (weak) Carathéodory conditions for multivalued mappings $F(t,x)$ and the notion of multivalued superposition operator.

**Definition (6.41).** A multivalued mapping $F$ over a Cartesian product of a measure space $(T,A)$ and a topological space $X$ is said to be a Carathéodory (resp. lower Carathéodory, upper Carathéodory) mapping if:

(a) For almost all $t \in T$, the mapping $F(t,\cdot) : X \to Y$ is continuous (respectively, lower semicontinuous, upper semicontinuous); and

(b) For all $x \in X$, the mapping $F(\cdot,x) : T \to Y$ is measurable.

The standard area of use of this definition is the case when $T$ is a segment on the real line $\mathbb{R}$ with the Lebesgue measure, $X$ is $\mathbb{R}^m$, and $Y$ is $\mathbb{R}^n$. The superposition operator in the singlevalued case is called Nemitsky operator and for a given mapping $f : T \times X \to Y$ it associates to every $\varphi : T \to X$ the composition mapping $t \mapsto f(t,\varphi(t))$ from $T$ into $Y$. Hence, Nemitsky operator acts from a space of mappings from $T$ into $X$ into a space of mappings from $T$ into $Y$. A typical problem is to find conditions for $f$ which are sufficient for the Nemitsky operator to map a prescribed space $S_1$ of mappings from $T$ into $X$ into another prescribed space $S_2$ of mappings from $T$ into $Y$ (see, e.g. [12]). A similar question can formally be stated in the multivalued case. However, the situation becomes more complicated. For multivalued Carathéodory mappings, Nemitsky (or superposition) operator associates to every continuous singlevalued mapping $g : T \to X$ (i.e., $S_1 = C(T,X)$) the set of all measurable selections of the mapping $\Phi(t) = F(t,g(t)), \Phi : T \to Y$.

**Theorem (6.42) [12].** Let $F : [t_1,t_2] \times \mathbb{R}^m \to \mathbb{R}^n$ be a compact-valued Carathéodory mapping and $G : [t_1,t_2] \to \mathbb{R}^m$ a compact-valued measurable mapping. Then $\Phi : [t_1,t_2] \to \mathbb{R}^n$, defined by $\Phi(t) = F(t,G(t))$, is measurable, i.e. the compact-valued Carathéodory mapping is superpositionally measurable (su-measurable).

The special case of this theorem with $G : [t_1,t_2] \to \mathbb{R}^m$ a singlevalued continuous mapping tells us that the superposition operator $N_F : C([t_1,t_2];\mathbb{R}^m) \to \mathcal{M}([t_1,t_2];\mathbb{R}^n)$ has nonempty values in the space $\mathcal{M}([t_1,t_2],\mathbb{R}^n)$ of all singlevalued measurable mappings from the segment $[t_1,t_2]$ into $\mathbb{R}^n$. In fact, the mapping $\Phi(t) = F(t,g(t))$ is measurable, for every $g \in C([t_1,t_2],\mathbb{R}^m)$ and hence admits a singlevalued measurable selection, according to Measurable selection theorem (6.12). Theorem (6.42) does not hold for upper Carathéodory compact-valued mappings (see [318]). It is a very useful fact that singlevalued measurable selections of the mapping $\Phi(t) = F(t,g(t))$ do exist for every upper Carathéodory mapping $F$. $\Phi$ merely admits a measurable compact-valued selection for which Theorem (6.12) is applicable.
Theorem (6.43) [58]. Let $F : [t_1, t_2] \times \mathbb{R}^m \to \mathbb{R}^n$ be a compact-valued upper Carathéodory mapping and $g : [t_1, t_2] \to \mathbb{R}^m$ a measurable singlevalued mapping. Then $\Phi : [t_1, t_2] \to \mathbb{R}^n$, defined by $\Phi(t) = F(t, g(t))$, admits a compact-valued measurable selection.

Clearly, one can generalize this theorem by assuming that $G : [t_1, t_2] \to \mathbb{R}^m$ is closed-valued and measurable: it suffices to consider a singlevalued measurable selection $g$ of $G$ and use Theorem (6.43).

Under the additional assumption that $F$ is integrally bounded we conclude that the superposition operator $N_F$ maps $C([t_1, t_2], \mathbb{R}^m)$ into the Banach space $L_1([t_1, t_2], \mathbb{R}^n)$.

Definition (6.44). A multivalued mapping $F : [t_1, t_2] \times U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$ is said to be integrally bounded if $\|F(t, x)\| = \sup\{|y| \mid y \in F(t, x)\} \leq \alpha(t) + \beta(t)\|x\|$, for some summable functions $\alpha, \beta \in L_1([t_1, t_2]; \mathbb{R})$ and for all $(t, x) \in [t, t_2] \times U$.

For a bounded $U \subset \mathbb{R}^m$ this definition implies Definition (6.36)(c) with

$$\|F(t, x)\| \leq \ell(t),$$

for some $\ell \in L_1([t_1, t_2]; \mathbb{R})$.

Theorem (6.45) [39]. Let $F : [t_1, t_2] \times \mathbb{R}^m \to \mathbb{R}^n$ be a compact-valued upper Carathéodory and integrally bounded mapping, $g : [t_1, t_2] \to \mathbb{R}^m$ continuous, and $\Phi : [t_1, t_2] \to \mathbb{R}^n$ defined by $\Phi(t) = F(t, g(t))$. Then every measurable selection $\varphi$ of $\Phi$ is a summable mapping, i.e. $\varphi \in L_1([t_1, t_2], \mathbb{R}^n)$. Hence, the superposition operator $N_F$ is the multivalued mapping from $C([t_1, t_2], \mathbb{R}^m)$ into $L_1([t_1, t_2], \mathbb{R}^n)$.

Theorem (6.46) [39]. Under the hypotheses of Theorem (6.45) let $F$ be a convex-valued mapping. Then the superposition operator $N_F : C([t_1, t_2], \mathbb{R}^m) \to L_1([t_1, t_2], \mathbb{R}^n)$ is a closed mapping with closed convex values.

Here, the closedness of a multivalued mapping means the closedness of the graph of this mapping. In addition to Theorem (6.46), every composition $T \circ N_F$ is a closed mapping whenever $T : L_1([t_1, t_2], \mathbb{R}^n) \to B$ is a continuous linear operator in a Banach space $B$.

Theorem (6.47) [22]. Let $\Pi = [t_0, t_0 + a] \times \text{Cl}(D(x_0, b) \subset \mathbb{R} \times \mathbb{R}^n$ and let $F : \Pi \to \mathbb{R}^n$ be an upper Carathéodory mapping, integrally bounded mapping with compact convex values. Then for some $0 < d \leq a$ on the segment $[t_0, t_0 + d]$, there exists a solution $x(\cdot)$ of the problem $x' \in F(t, x), x(t_0) = x_0$.

Recall, that the term “$x(\cdot)$ is a solution on the segment $[t_0, t_0 + d]$ of the problem $x' \in F(t, x)$, $x(t_0) = x_0$” means that $x(\cdot)$ is an absolutely continuous mapping such that $(t, x(t)) \in \Pi$ for all $t \in [t_0, t_0 + d]$; $x(t_0) = x_0$; and $x'(t) \in F(t, x(t))$, for almost all $t \in [t_0, t_0 + d]$. 

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Proof.

I. Construction

Let:

(1) $F$ be integrally bounded by mappings $\alpha, \beta \in L_1([t_0, t_0 + d], \mathbb{R}^n)$, i.e. 
$$\|F(t, x)\| \leq \alpha(t) + \beta(t)\|x\|;$$ and

(2) $m(t) = \int_{t_0}^{t}(\alpha(\tau) + (\|x_0\| + b)\beta(\tau)) \, d\tau.$

We claim that then:

(a) $m(\cdot)$ is a continuous function with $m(t_0) = 0$; and
(b) There exists $0 < d \leq a$ such that $m(t_0 + d) \leq b$. 

To construct a solution on the segment $[t_0, t_0 + d]$, let:

(3) $N_F : C([t_0, t_0 + d], \mathbb{R}^n) \to L_1([t_0, t_0 + d], \mathbb{R}^n)$ be the superposition operator defined by $F$;

(4) For every $g \in C([t_0, t_0 + d], \mathbb{R}^n)$, the subset $A_F(g)$ of $C$ be defined by setting:

$$[A_F(g)](t) = x_0 + \{ \int_{t_0}^{t} \varphi(\tau) \, d\tau \mid \varphi \in N_F(g) \};$$ and

(5) $D = \text{Cl}(D(x_0, b))$ be the closed $b$-ball in $C$ centered at the point $x_0(t) \equiv x_0$.

We claim that then:

(c) $A_F : C \to C$ is a closed mapping with closed convex values;

(d) $\text{Cl}(A_F(D))$ is compact in $C$, i.e. $A_F$ is compact operator;

(e) $A_F(D) \subset D$ and $A_F|_D$ is upper semicontinuous;

(f) $A_F$ has a fixed point $x(\cdot) \in C$, i.e. $x(\cdot) \in A_F(x(\cdot))$; and

(g) Such a fixed point is the desired solution of the problem $x' \in F(t, x)$, $x(t_0) = x_0$ on the segment $[t_0, t_0 + d]$.

II. Verification

(a), (b): Obvious.

(c) Corollary of Theorem (6.46) and linearity of the Lebesgue integral.

(d) If $g \in C$ with $\|g - \bar{x}_0\| \leq b$ and $h \in A_F(g)$, then

$$h(t) = x_0 + \int_{t_0}^{t} \varphi(\tau) \, d\tau \text{ for some } \varphi \in N_F(g),$$

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i.e. \( \varphi(\tau) \in F(\tau, g(\tau)) \). Hence

\[
\|h(t)\|_{\mathbb{R}^n} \leq \|x_0\| + \int_{t_0}^{t} \|\varphi(\tau)\| \, d\tau \leq \|x_0\| + \int_{t_0}^{t} (\alpha(\tau) + (\|x_0\| + b)\beta(\tau)) \, d\tau \leq \|x_0\| + m(t) \leq \|x_0\| + m(t_0 + d) \leq \|x_0\| + b,
\]

i.e. \( \|h\|_{C} \leq \|x_0\| + b \). By virtue of the Arzela-Ascoli theorem we only need to check that \( A_F(D) \) is an equicontinuous family of mappings. Using our previous notations, we have

\[
\|h(t') - h(t'')\|_{\mathbb{R}^n} = \left\| \int_{t'}^{t''} \varphi(\tau) \, d\tau \right\|_{\mathbb{R}^n} \leq \int_{t'}^{t''} \alpha(\tau) \, d\tau + (\|x_0\| + b) \int_{t'}^{t''} \beta(\tau) \, d\tau
\]

and therefore the statement follows by the absolute continuity of the Lebesgue integral.

(e) With the notations from the proof of (d), we have for \( g \in C \) with \( \|g - \bar{x}_0\| \leq b \) and for \( h \in A_F(g) \):

\[
\|h - \bar{x}_0\|_{C} = \max \{ \|h(t) - x_0\|_{\mathbb{R}^n} \mid t \in [t_0, t_0 + d] \} = \max \{ \|t \int_{t_0}^{t} \varphi(\tau) \, d\tau\|_{\mathbb{R}^n} \mid t \in [t_0, t_0 + d] \} \leq m(t_0 + d) \leq b.
\]

Hence \( A_F(D) \subset D \). The upper semicontinuity of \( A_F|_D \) follows by its closedness (in the sense that the graph of \( A_F \) is closed, see (c)), and from its compactness (in the sense that the image of a bounded set has a compact closure, see (d)).

(f) An application of the Brouwer-Kakutani fixed-point principle (see e.g. [109]) to the mapping \( A_F \).

(g) Evident. Theorem (6.47) is thus proved. ■

Observe, that the operator \( A_F : C \to C \), defined in (4) of the proof of Theorem (6.47), is often called a multivalued integral of the superposition operator \( N_F \). More generally, the integral \( \int_{T} F(t) \, dt \) of a multivalued mapping \( F \) from a measurable space \( T \) into \( \mathbb{R}^n \) is usually defined as the set \( \{ \int_{T} f(t) \, dt \} \) of all integrals of all integrable selections \( f \) of \( F \), see [18].

We finish this chapter by an observation that the Carathéodory conditions and Measurable selection theorem imply the well-known Filippov implicit function theorem (lemma).
Theorem (6.48) [134]. Let $F : [t_1, t_2] \times \mathbb{R}^m \to \mathbb{R}^n$ be a compact-valued Carathéodory mapping and $G : [t_1, t_2] \to \mathbb{R}^m$ a compact-valued measurable mapping. Then for every singlevalued measurable selection $\varphi(\cdot)$ of the composition $\Phi(\cdot) = F(\cdot, G(\cdot))$, there exists a singlevalued measurable selection $g$ of $G$ such that $\varphi(\cdot)$ is “almost all” selection of the composition $F(\cdot,g(\cdot))$, i.e. $\varphi(t) \in F(t,g(t))$, for almost all $t$.

For a singlevalued $f = F$, this theorem states the possibility of a singlevalued solution of the inclusion $\varphi(t) \in f(t,G(t))$, with respect to the second variable of the mapping $f$, i.e. we “implicitly express” $g(t)$ via $\varphi(t)$, for almost all $t \in [t_1, t_2]$. The proof of Theorem (6.48) is based on the consideration of the intersection of the mapping $G$ with the mapping $H : [t_1, t_2] \to \mathbb{R}^m$, defined as

$$H(t) = \{x \in \mathbb{R}^m \mid \varphi(t) \in F(t,x)\}.$$ 

Clearly, $G(t) \cap H(t)$ are nonempty compacta in $\mathbb{R}^m$, $t \in [t_1, t_2]$ and every measurable selection of $G \cap H$ is the desired selection $g$ of $G$, by Theorem (6.12).

Only one point must be verified: the measurability of $G \cap H$. This can be done using the fact that the compact-valued Carathéodory mapping has the so-called Scorza-Dragoni property – a multivalued analogue of the well-known Lusin property. For a metric space $M$ with a $\sigma$-measure on the Borel subsets a multivalued mapping $F : M \times \mathbb{R}^m \to \mathbb{R}^n$ is said to have the upper (resp. lower) Scorza-Dragoni property if for a given $\delta > 0$, one can find a closed subset $M_\delta \subset M$ with $\mu(M \setminus M_\delta) < \delta$ and the restriction $F$ to $M_\delta \times \mathbb{R}^m$ is upper (resp. lower) semicontinuous. $F$ has the Scorza-Dragoni property if $F$ has both upper and lower Scorza-Dragoni property.

Theorem (6.49) [203]. For a compact-valued mapping $F : M \times \mathbb{R}^m \to \mathbb{R}^n$ the following assertions are equivalent:

1. $F$ is a Carathéodory mapping; and
2. $F$ has the Scorza-Dragoni property.