The Bryant–Ferry–Mio–Weinberger construction of generalized manifolds

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Following Bryant, Ferry, Mio and Weinberger we construct generalized manifolds as limits of controlled sequences \( \{ X_i \to X_{i-1} : i = 1, 2, \ldots \} \) of controlled Poincaré spaces. The basic ingredient is the \( \varepsilon-\delta \)–surgery sequence recently proved by Pedersen, Quinn and Ranicki. Since one has to apply it not only in cases when the target is a manifold, but a controlled Poincaré complex, we explain this issue very roughly. Specifically, it is applied in the inductive step to construct the desired controlled homotopy equivalence \( p_{i+1} : X_{i+1} \to X_i \). Our main theorem requires a sufficiently controlled Poincaré structure on \( X_i \) (over \( X_{i-1} \)). Our construction shows that this can be achieved. In fact, the Poincaré structure of \( X_i \) depends upon a homotopy equivalence used to glue two manifold pieces together (the rest is surgery theory leaving unaltered the Poincaré structure). It follows from the \( \varepsilon-\delta \)–surgery sequence (more precisely from the Wall realization part) that this homotopy equivalence is sufficiently well controlled. In the final section we give additional explanation why the limit space of the \( X_i \)’s has no resolution.

57PXX; 55RXX

1 Preliminaries

A generalized \( n \)–dimensional manifold \( X \) is characterized by the following two properties:

(i) \( X \) is a Euclidean neighborhood retract (ENR); and

(ii) \( X \) has the local homology (with integer coefficients) of the Euclidean \( n \)–space \( \mathbb{R}^n \), ie

\[
H_*(X, X \setminus \{ x \}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{ 0 \}).
\]

Since we deal here with locally compact separable metric spaces of finite (covering) dimension, ENRs are the same as ANRs.

Generalized manifolds are Poincaré spaces, in particular they have the Spivak normal fibrations \( v_X \). The total space of \( v_X \) is the boundary of a regular neighborhood.
$N(X) \subset \mathbb{R}^L$ of an embedding $X \subset \mathbb{R}^L$, for some large $L$. One can assume that $N(X)$ is a mapping cylinder neighborhood (see Lacher [5, Corollary 11.2]).

The global Poincaré duality of Poincaré spaces does not imply the local homology condition (ii) above. The local homology condition can be understood as the “controlled” global Poincaré duality (see Quinn [9, p270], and Bryant–Ferry–Mio–Weinberger [1, Proposition 4.5]). More precisely, one has the following:

**Theorem 1.1** Let $X$ be a compact ANR Poincaré duality space of finite (covering) dimension. Then $X$ is a generalized manifold if and only if for every $\delta > 0$, $X$ is a $\delta$–Poincaré space (over $X$).

The definition of the $\delta$–Poincaré property is given below. The following basic fact about homology manifolds was proved by Ferry and Pedersen [4, Theorem 16.6].

**Theorem 1.2** Let $X$ be an ANR homology manifold. Then $\nu_X$ has a canonical TOP reduction.

This statement is equivalent to existence of degree-one normal maps $f : M^n \to X$, where $M^n$ is a (closed) topological $n$–manifold, hence the structure set $\mathcal{S}^{\text{TOP}}(X)$ can be identified with $[X, G/\text{TOP}]$.

Let us denote the 4–periodic simply connected surgery spectrum by $\mathbb{L}$ and let $\widehat{\mathbb{L}}$ be the connected covering of $\mathbb{L}$. There is a (canonical) map of spectra $\widehat{\mathbb{L}} \to \mathbb{L}$ given by the action of $\widehat{\mathbb{L}}$ on $\mathbb{L}$. Note that $\widehat{\mathbb{L}}_0$ is $G/\text{TOP}$.

If $M^n$ is a topological manifold there exists a fundamental class $[M]_\mathbb{L} \in H_n(M; \mathbb{L}^*)$, where $\mathbb{L}^*$ is the symmetric surgery spectrum (see Ranicki [11, Chapters 13 and 16]).

**Theorem 1.3** If $M^n$ is a closed oriented topological $n$–manifold, then the cap product with $[M]_\mathbb{L}$ defines a Poincaré duality of $\mathbb{L}$–(co)homology

$$H^p(M; \mathbb{L}) \cong H_{n-p}(M; \mathbb{L})$$

and $\widehat{\mathbb{L}}$–(co)homology

$$H^p(M; \widehat{\mathbb{L}}) \cong H_{n-p}(M; \widehat{\mathbb{L}}).$$

Since $H^0(M; \mathbb{L}) = [M, \mathbb{Z} \times G/\text{TOP}]$ and $H^0(M; \widehat{\mathbb{L}}) = [M, G/\text{TOP}]$, we have

$$H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$$

and the map $\widehat{\mathbb{L}} \to \mathbb{L}$ has the property that the image of

$$H_n(M; \widehat{\mathbb{L}}) \to H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$$
We assume that $M$ is a PL manifold, or that $M$ has a cell structure. The $X_i$ are built by gluing manifolds along boundaries with homotopy equivalences, and by doing some surgeries outside the singular sets. Hence all the $X_i$ have cell decompositions.

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Beginning with a closed topological $n$–manifold $M^n$, for $n \geq 5$, and $\sigma \in H_n(M; \mathbb{L})$, we shall construct a sequence of closed Poincaré complexes $X_0, X_1, X_2, \ldots$, and maps $p_i: X_i \to X_{i-1}$ and $p_0: X_0 \to M$.

We assume that $M$ is a PL manifold, or that $M$ has a cell structure. The $X_i$ are built by gluing manifolds along boundaries with homotopy equivalences, and by doing some surgeries outside the singular sets. Hence all the $X_i$ have cell decompositions.
We can assume that the \( X_i \) lie in a (large enough) Euclidean space \( \mathbb{R}^L \) which induces the metric on \( X_i \). So the cell chain complex \( C_\#(X_i) \) can be considered as a geometric chain complex over \( X_{i-1} \) with respect to \( p_i: X_i \to X_{i-1} \), i.e. the distance between two cells of \( X_i \) over \( X_{i-1} \) is the distance between the images of the centers of these two cells in \( X_{i-1} \). Let us denote the distance function by \( d \).

We now list five properties of the sequence \( \{(X_i, p_i)\}_{i=0} \), including some definitions and comments. For each \( i \geq 0 \) we choose positive real numbers \( \xi_i \) and \( \eta_i \).

(i) \( p_i: X_i \to X_{i-1} \) and \( p_0: X_0 \to M \) are \( UV^1 \)-maps. This means that for every \( \varepsilon > 0 \) and for all diagrams

\[
\begin{array}{ccc}
K_0 & \overset{\alpha_0}{\longrightarrow} & X_i \\
\downarrow & \searrow & \downarrow p_i \\
K & \underset{\alpha}{\longrightarrow} & X_{i-1}
\end{array}
\]

with \( K \) a 2–complex, \( K_0 \subset K \) a subcomplex and maps \( \alpha_0, \alpha \), there is a map \( \alpha \) such that \( \underline{\alpha}|_{K_0} = \alpha_0 \) and \( d(p_i \circ \alpha, \alpha) < \varepsilon \). (This is also called \( UV^1(\varepsilon) \) property.)

(ii) \( X_i \) is an \( \eta_i \)--Poincaré complex over \( X_{i-1} \), i.e.

(a) all cells of \( X_{i-1} \) have diameter \( < \eta_i \) over \( X_{i-1} \); and

(b) there is an \( n \)--cycle \( c \in C_n(X_i) \) which induces an \( \eta_i \)--chain equivalence \( \cap_c: C_\#(X_i) \to C_{n-\#}(X_i) \).

Equivalently, the diagonal \( \Delta_\#(c) = \sum c' \otimes c'' \in C_\#(X) \otimes C_\#(X) \) has the property that \( d(c', c'') < \eta_i \) for all tensor products appearing in \( \Delta_\#(c) \).

(iii) \( p_i: X_i \to X_{i-1} \) is an \( \xi_i \)--homotopy equivalence over \( X_{i-2} \), for \( i \geq 2 \). In other words, there exist an inverse \( p'_i: X_{i-1} \to X_i \) and homotopies \( h_i: p'_i \circ p_i \simeq \text{Id}_{X_i} \) and \( h'_i: p_i \circ p'_i \simeq \text{Id}_{X_{i-1}} \) such that the tracks

\[
\left\{(p_{i-1} \circ p_i \circ h_i)(x, t) : t \in [0, 1]\right\} \quad \text{and} \quad \left\{(p_i \circ h'_i)(x', t) : t \in [0, 1]\right\}
\]

have diameter less than \( \xi_i \), for each \( x \in X_i \) (respectively, \( x' \in X_{i-1} \)). Note that \( p_0 \) need not be a homotopy equivalence.

(iv) There is a regular neighborhood \( W_0 \subset \mathbb{R}^L \) of \( X_0 \) such that \( X_i \subset W_0 \), for \( i = 0, 1, \ldots \), and retractions \( r_i: W_0 \to X_i \), satisfying \( d(r_i, r_{i-1}) < \xi_i \) in \( \mathbb{R}^L \).

(v) There are “thin” regular neighborhoods \( W_i \subset \mathbb{R}^L \) with \( \pi_i: W_i \to X_i \), where \( W_i \subset W_{i-1} \) such that \( W_{i-1} \setminus W_i \) is an \( \xi_i \)--thin \( h \)--cobordism with respect to \( r_i: W_0 \to X_i \).

Let \( W = W_{i-1} \setminus W_i \). Then there exist deformation retractions \( r^0_i: W \to \partial_0 W \) and \( r^1_i: W \to \partial_1 W \) with tracks of size \( < \xi_i \) over \( X_{i-1} \), i.e. the diameters of...
\[(r_i \circ r_i^0)(w) : t \in [0,1]\] and \[(r_i \circ r_i^1)(w) : t \in [0,1]\] are smaller than \(\xi_i\). Moreover, we can choose \(\eta_i\) and \(\xi_i\) such that
(a) \(\sum \eta_i < \infty\); and
(b) \(W_{i-1} \setminus W_i\) has a \(\delta_i\)-product structure with \(\sum \delta_i < \infty\), ie there is a homeomorphism
\[W = W_{i-1} \setminus W_i \looparrowright_{\partial_0 W} \partial_0 W \times I\]
satisfying
\[\operatorname{diam}\{ (r_i \circ H)(w,t) : t \in I \} < \delta_i,\]
for every \(w \in \partial_0 W\).

The property (v)(b) above follows from the “thin \(h\)-cobordism” theorem (see the article [8] by Quinn). One can assume that \(\sum \xi_i < \infty\). Let \(X = \bigcap_i W_i\). We are going to show that \(X\) is a generalized manifold:

1. The map \(r = \lim_{i \to \infty} r_i : W_0 \to X\) is well–defined and is a retraction, hence \(X\) is an ANR.
2. To show that \(X\) is a generalized manifold we shall apply the next two theorems. They also imply Theorem 1.1 above. The first one is due to Daverman and Husch [2], but it is already indicated in [8] (see the remark after Theorem 3.3.2).

**Theorem 2.1** Suppose that \(M^n\) is a closed topological \(n\)-manifold, \(B\) is an ANR, and \(p : M \to B\) is proper and onto. Then \(B\) is a generalized manifold, provided that \(p\) is an approximate fibration.

Approximate fibrations are characterized by the property that for every \(\varepsilon > 0\) and every diagram
\[
\begin{array}{ccc}
K \times \{0\} & \xrightarrow{H_0} & M \\
\downarrow & & \downarrow p \\
K \times I & \xrightarrow{H} & B
\end{array}
\]
where \(K\) is a polyhedron, there exists a lifting \(H\) of \(h\) such that \(d(p \circ H, h) < \varepsilon\). Here \(d\) is a metric on \(B\). In other words, \(p : M \to B\) has the \(\varepsilon\)-homotopy lifting property for all \(\varepsilon > 0\).

We apply Theorem 2.1 to the map \(\rho : \partial W_0 \to X\) defined as follows: Let \(\rho : W_0 \to X\) be the map which associates to \(w \in W_0\) the endpoint \(\rho(w) \in X\) following the tracks defined by the “thin” product structures of the \(h\)-cobordism when decomposing
\[W_0 = (W_0 \setminus \overset{\circ}{W}_1) \cup (W_1 \setminus \overset{\circ}{W}_2) \cup \ldots\]
The restriction to $\partial W_0$ will also be denoted by $\rho$. By (v)(b) above, the map $\rho$ is well-defined and continuous. We will show that it is an $\varepsilon$–approximate fibration for all $\varepsilon > 0$.

The map $\rho: W_0 \to X$ is the limit of maps $\rho_i: W_0 \to X_i$, where $\rho_i$ is the composition given by the tracks $(W_0 \setminus W_1) \cup (W_1 \setminus W_2) \cup \cdots \cup (W_{i-1} \setminus W_i)$ followed by $\pi_i: W_i \to X_i$. The second theorem is due to Bryant, Ferry, Mio and Weinberger [1, Proposition 4.5].

**Theorem 2.2** Given $n$ and $B$, there exist $\varepsilon_0 > 0$ and $T > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following holds: If $X \xrightarrow{p} B$ is an $\varepsilon$–Poincaré complex with respect to the $\mathcal{U}V^1$–map $p$ and $W \subset \mathbb{R}^L$ is a regular neighborhood of $X \subset \mathbb{R}^L$, ie $\pi: W \to X$ is a neighborhood retraction, then $\pi|_{\partial W}: \partial W \to X$ has the $T\varepsilon$–lifting property, provided that the codimension of $X$ in $\mathbb{R}^L$ is $\geq 3$.

This is applied as follows: Let $B \subset \mathbb{R}^L$ be a (small) regular neighborhood of $X \subset \mathbb{R}^L$. Hence $X_k \subset W_k \subset B$ for sufficiently large $k$. It follows by property (ii) that $X_i$ is an $\eta_i$–Poincaré complex over $X_i \xrightarrow{p_i} X_{i-1} \subset B$, hence (for $i$ sufficiently large) we get the following:

**Corollary 2.3** $\rho_i: \partial W_0 \to X_i$ is a $T\eta_i$–approximate fibration over $B$.

**Proof** By the theorem above, $\pi_i: \partial W_i \to X_i$ is a $T\eta_i$–approximate fibration over $B$, hence so is $\rho_i: \partial W_0 \cong \partial W_i \to X_i$. \hfill $\Box$

It follows by construction that $\lim X_i = X \subset B$, and so we have, in the limit, an approximate fibration $\rho: \partial W_0 \to X \xrightarrow{p_i} X_i \xrightarrow{} X$, ie $X$ is a generalized manifold. We will show in Section 4 that $I(X)$ is determined by the $\mathcal{Z}$–sector of $\sigma \in H_n(M; \mathbb{L})$.

### 3 Construction of the sequence of controlled Poincaré complexes

Before we begin with the construction we need more fundamental results about controlled surgery and approximations.
3.1 $\varepsilon-\delta$ surgery theory

We recall the main theorem of the article [7] by Pedersen, Quinn and Ranicki. Let $B$ be a finite–dimensional compact ANR, and $N^n$ a compact $n$–manifold (possibly with nonempty boundary $\partial N$), where $n \geq 4$. Then there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exist $\delta > 0$ with the following property:

If $p: N \to B$ is a $\mathcal{U}V^1(\delta)$ map, then there exists a controlled exact surgery sequence

\[ H_{n+1}(B; \mathbb{L}) \to S_{\varepsilon,\delta}(N, p) \to [N, \partial N; G/\text{TOP}, \ast] \xrightarrow{\Theta} H_n(B; \mathbb{L}). \]

The controlled structure set $S_{\varepsilon,\delta}(N, p)$ is defined as follows. Elements of $S_{\varepsilon,\delta}(N, p)$ are (equivalence) classes of $(M, g)$, where $M$ is an $n$–manifold, $g: M \to N$ is a $\delta$–homotopy equivalence over $B$ and $g|_{\partial M}: \partial M \to \partial N$ is a homeomorphism. The pair $(M, g)$ is related to $(M', g')$ if there is a homeomorphism $h: M \to M'$, such that the diagram

\[ \begin{array}{ccc}
\partial M & \xrightarrow{h} & \partial M' \\
\downarrow g & & \downarrow g' \\
\partial N & \xrightarrow{g'} & \partial N \\
\end{array} \]

commutes, and $g' \circ h$ is $\varepsilon$–homotopic to $g$ over $B$. Since $\varepsilon$ is fixed, this relation is not transitive. It is part of the assertion that it is actually an equivalence relation. Then $S_{\varepsilon,\delta}(N, p)$ is the set of equivalence classes of pairs $(M, g)$.

As in the classical surgery theory, the map

\[ H_{n+1}(B; \mathbb{L}) \to S_{\varepsilon,\delta}(N, p) \]

is the controlled realization of surgery obstructions, and

\[ S_{\varepsilon,\delta}(N, p) \to [N, \partial N; G/\text{TOP}, \ast] \xrightarrow{\Theta} H_n(B; \mathbb{L}) \]

is the actual (controlled) surgery part. The following discussion will show that (3) also holds for controlled Poincaré spaces (see Theorem 3.1 below). Moreover, $\delta$ is also of (arbitrary) small size, provided that such is also $\varepsilon$.

To see this we will go through some of the main points of the proof of [7, Theorem 1]. For $\eta, \eta' > 0$ we denote by $L_n(B, \mathbb{Z}, \eta, \eta')$ the set of highly $\eta$–connected $n$–dimensional quadratic Poincaré complexes modulo highly $\eta'$–connected algebraic cobordisms. Then there is a well–defined obstruction map

\[ \Theta_{\eta}: [N, G/\text{TOP}] \to L_n(B, \mathbb{Z}, \eta, \eta') \]
Theorem 1.1. \[
\text{will follow by the Squeezing Lemma of (4) to be exact, ie that the composition map is zero, (4) will be completed by showing that the assembly map from (4) above, with a sufficiently small } \delta \text{ we want a splitting where each piece is } \delta \text{-controlled Poincaré complexes over } B. \text{ By Theorem 1.1 above, this holds in particular for generalized manifolds.}
\]

Given \( \eta > 0 \) there is an \( \eta' > 0 \) such that if \( (f', b') \) and \( (f'', b'') \) are normally bordant, highly \( \eta \)-connected, degree-one, normal maps, there is then a highly \( \eta' \)-connected normal bordism between them. (Again this is true if \( N \) is an \( \eta \)-Poincaré complex over \( B \).) This defines an element of \( S_{\epsilon, \delta}(N, p) \), and shows the semi–exactness of the sequence

\[
S_{\epsilon, \delta}(N, p) \to [N, G/TOP] \to L_n(B, \mathbb{Z}, \eta, \eta').
\]

One cannot expect the sequence (4) to be exact, ie that the composition map is zero, since passing from topology to algebra one loses control. As it was noted by Pedersen, Quinn and Ranicki [7, p243], \( \epsilon \) and \( \delta \) are determined by the controlled Hurewicz and Whitehead theorems. Exactness of (4) will follow by the Squeezing Lemma of Pedersen and Yamasaki [6, Lemma 4].

The proof of (3) will be completed by showing that the assembly map

\[
A: H_n(B; \mathbb{L}) \to L_n(B, \mathbb{Z}, \eta, \eta')
\]

is bijective for sufficiently small \( \eta \). This follows by splitting the controlled quadratic Poincaré complexes (ie the elements of \( L_n(B', \mathbb{Z}, \eta, \eta') \)) into small pieces over small simplices of \( B \) (we assume for simplicity that \( B \) is triangulated). If \( \delta \) is given, and if we want a splitting where each piece is \( \delta \)-controlled, we must start the subdivision with a sufficiently small \( \eta \)-controlled quadratic Poincaré complex (see the following Remark). This can be done by [7, Lemma 6] (see also Yamasaki [12, Lemma 2.5]). Since \( A \circ \Theta = \Theta_\eta \), we get (3) from (4). The stability constant \( \epsilon_0 \) is determined by the largest \( \eta \) for which \( A \) is bijective.
Remark Yamasaki has estimated the size of $\eta$ in the Splitting Lemma. If one performs a splitting so that the two summands are $\delta$–controlled, then one needs an $\eta$–controlled algebraic quadratic Poincaré complex with $\eta$ of size $\delta/(an^k + b)$, where $a, b, k$ depend on $X$ ($k$ is conjectured to be 1), and $n$ is the length of the complex. Of course, squeezing also follows from the bijectivity of $A$ for small $\eta$, but the result [6, Lemma 3] of Pedersen and Yamasaki is somehow a clean statement to apply (see Theorem 3.1 below). We also note that the bijectivity of $A$ is of course, independent of whether $N$ is a manifold or a Poincaré complex.

Theorem 3.1 Suppose that $N \xrightarrow{p} B$ is a $U^1$ map. Let $\delta > 0$ be given (sufficiently small, ie $\delta < \delta_0$ for some $\delta_0$). Then there is $\eta > 0$ (small with respect to $\delta$), such that if $N$ is an $\eta$–Poincaré complex over $B$, and $(f, b): M \to N$ is a degree-one normal map, then $\Theta(f, b) = 0 \in H_n(B; \mathbb{L})$ if (and only if) $(f, b)$ is normally bordant to a $\delta$–equivalence.

The “only if” part is more delicate and follows by [6, Lemma 3]. So let $f: M^n \to N^n$ be a $\delta$–equivalence defining a quadratic $\eta_1$–Poincaré complex $C$ in $L_n(B, \mathbb{Z}, \eta_1, \eta'_1)$ which is $\eta_1$–cobordant to zero via $[N, G/TOP] \to L_n(B, \mathbb{Z}, \eta_1, \eta'_1)$.

Then $C$ is $\kappa \eta_1$–cobordant to an arbitrary small quadratic Poincaré complex (ie to a quadratic $\eta$–complex) which is $\kappa \eta'_1$–cobordant to zero, with $\eta_1$ sufficiently small (ie $\eta$ sufficiently small). In this case we can also assume that $A$ is bijective. This proves the “only if” part.

Theorem 3.1 can also be stated as follows:

Theorem 3.1’ Let $N$ be a sufficiently fine $\eta$–Poincaré complex over a $U^1$–map $p: N \to B$. Then there exist $\epsilon > 0$ and $\delta > 0$, both sufficiently small, such that the sequence

$$S_{\epsilon, \delta}(N, p) \to [N, G/TOP] \to H_n(B; \mathbb{L})$$

is exact. In particular, it holds for generalized manifolds.

3.2 $U^1$ approximation

Here we recall the results [1, Proposition 4.3, Theorem 4.4] of Bryant, Ferry, Mio and Weinberger.

Theorem 3.2 Suppose that $f: (M^n, \partial M) \to B$ is a continuous map from a compact $n$–manifold with boundary such that the homotopy fiber of $f$ is simply connected. If $n \geq 5$ then $f$ is homotopic to a $U^1$–map. In case that $f|_{\partial M}$ is already $U^1$, the homotopy is relative $\partial M$. 

We state the second theorem in the form which we will need.

**Theorem 3.3** (Ferry [3, Theorem 10.1]) Let $p: N^n \to B$ be a map from a compact $n$–manifold into a polyhedron, where $n \geq 5$. Then:

(i) Given $\varepsilon > 0$, there is a $\delta > 0$, such that if $p$ is a $\mathcal{U}V^1(\delta)$–map then $p$ is $\varepsilon$–homotopic to a $\mathcal{U}V^1$–map.

(ii) Suppose that $p: N \to B$ is a $\mathcal{U}V^1$ map. Then for each $\varepsilon > 0$ there is a $\delta > 0$ (depending on $p$ and $\varepsilon$) such that if $f: M \to N$ is a $(\delta-1)$–connected map (over $B$) from a compact manifold $M$ of dimension at least 5, then $f$ is $\varepsilon$–close over $B$ to a $\mathcal{U}V^1$–map $g: M \to N$.

### 3.3 Controlled gluing

**Theorem 3.4** (Bryant–Ferry–Mio–Weinberger [1, Proposition 4.6]) Let $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ be (orientable) manifolds and $p_i: M_i \to B$ be $\mathcal{U}V^1$–maps. Then there exist $\varepsilon_0 > 0$ and $T > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$ and $h: \partial M_1 \to \partial M_2$ an (orientation preserving) $\varepsilon$–equivalence, $M_1 \cup_h M_2$ is a $T\varepsilon$–Poincaré complex over $B$.

### 3.4 Approximation of retractions

**Theorem 3.5** (Bryant–Ferry–Mio–Weinberger [1, Proposition 4.10]) Let $X$ and $Y$ be finite polyhedra. Suppose that $V$ is a regular neighborhood of $X$ with $\dim V \geq 2 \dim Y + 1$ and $r: V \to X$ is a retraction. If $f: Y \to X$ is an $\varepsilon$–equivalence with respect to $p: X \to B$, then there exists an embedding $i: Y \to V$ and a retraction $s: V \to i(Y)$ with $d(p \circ r, p \circ s) < 2\varepsilon$.

We now begin with the construction. Let $M^n$ be a closed oriented (topological) manifold of dimension $n \geq 6$. Let $\sigma \in H_n(M; \mathbb{L})$ be fixed. Moreover, we assume that $M$ is equipped with a simplicial structure. Then let $M = B \cup_D C$ be such that $B$ is a regular neighborhood of the $2$–skeleton, $D = \partial B$ is its boundary and $C$ is the closure of the complement of $B$. So $D = \partial C = B \cap C$ is of dimension $\geq 5$.

By Theorem 3.2 above we can replace $(B, D) \subset M$ and $(C, D) \subset M$, by $\mathcal{U}V^1$–maps $j: (B, D) \to M$ and $j: (C, D) \to M$, and realize $\sigma$ according to $\mathcal{H}_n(M; \mathbb{L}) \to S_{\varepsilon, \delta}(D, j)$ by a degree-one normal map $F_\sigma: V \to D \times I$ with $\partial_0 V = D$, $\partial_1 V = D'$, $F_\sigma|_{\partial_0 V} = \text{Id}$ and $F_\sigma|_{\partial_1 V} = D'$ a $\delta$–equivalence over $M$.

We then define $X_0 = B \cup_{f_\sigma} -V \cup_{\text{Id}} C$, where $-V$ is the cobordism $V$ turned upside down. We use the map $-F_\sigma \cup \text{Id}$, $-V \cup_{\text{Id}} C \to D \times I \cup C \cong C$ to extend $j$ to a map $p_0: X_0 \to M$. 

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The Wall realization $V \to D \times I$ is such that $V$ is a cobordism built from $D$ by adding high–dimensional handles (similarly beginning with $D'$). Therefore $p_0$ is a $\mathcal{U}V^1$ map: if $(K, L)$ is a simplicial pair with $K$ a 2–complex, and if there is given a diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\alpha_0} & X_0 \\
\downarrow & & \downarrow p_0 \\
K & \xrightarrow{\alpha} & M
\end{array}
$$

then we first move (by an arbitrary small approximation) $\alpha$ and $\alpha_0$ into $B$ by general position arguments. Then one uses the $\mathcal{U}V^1$–property of $j: B \to M$. By Theorem 3.4, $X_0$ is a $T\delta$–Poincaré complex over $M$. Note that we can choose $\delta$ as small as we want, hence we get an $\eta_0$–Poincaré complex for a prescribed $\eta_0$. This completes the first step.

To continue we define a manifold $M^n_0$ and a degree-one normal map $g_0: M^n_0 \to X_0$ by

$$
M_0 = B \cup_{i_1} V \cup_{i_2} C \to B \cup_{i_1} D \times I \cup_{j_0} -V \cup_{i_2} C \cong X_0,
$$

using $F_\sigma \cup \text{Id}: V \cup_{i_1} -V \to D \times I \cup_{j_0} -V$. By construction it has a controlled surgery obstruction $\sigma \in H_n(M; \mathbb{L})$.

Moreover, there is $\bar{\sigma} \in H_n(X_0; \mathbb{L})$ with $p_0^* (\bar{\sigma}) = \sigma$. This can be seen from the diagram

$$
\begin{array}{ccc}
H_n(M_0; \mathbb{L}) & \xrightarrow{g_0^*} & H_n(X_0; \mathbb{L}) \\
\downarrow & & \downarrow p_0^* \\
H^0(M_0; \mathbb{L}) & \xleftarrow{g_0^*} & H^0(X_0; \mathbb{L}) \\
\end{array}
$$

The vertical isomorphisms are Poincaré dualities. Since $p_0$ is a $\mathcal{U}V^1$ map, $\bar{\sigma}$ belongs to the same $\mathcal{L}$–sector as $\sigma$. We will again denote $\bar{\sigma}$ by $\sigma$.

We construct $p_1: X_1 \to X_0$ as above: Let $M_0 = B_1 \cup C_1$, let $B_1$ be a regular neighborhood of the 2–skeleton (as fine as we want), let $C_1$ be the closure of the complement and let $D_1 = C_1 \cap B_1 = \partial C_1 = \partial B_1$, and $g_0: D_1 \to X_0$ be a $\mathcal{U}V^1$ map. Then we realize $\sigma \in H_n(X_0; \mathbb{L}) \to S_{\epsilon_1, \delta_1}(D_1, g_0)$ by $F_{1, \sigma}: V_1 \to D_1 \times I$ with $\partial_0 V_1 = D_1$, $\partial_1 V_1 = D_1'$, $F_{1, \sigma} \mid_{\partial_0 V_1} = \text{Id}$ and $f_{1, \sigma} = F_{1, \sigma} \mid_{\partial_1 V_1}: D_1' \to D_1$ a $\delta_1$–equivalence over $X_0$.

We define $p'_1: X'_1 \to X_0$ by

$$
X'_1 = B_1 \cup_{f_{1, \sigma}} -V_1 \cup_{\text{Id}} C_1 \xrightarrow{f'_1} M_0 \cong B_1 \cup_{\text{Id}} D_1 \times I \cup_{\text{Id}} C_1.
$$
using \(-F_{1,\sigma} : -V_1 \to D_1 \times I\), and then \(p'_1 = g_0 \circ f'_1 : X'_1 \to M_0 \to X_0\).

We now observe that

(i) by Theorem 3.4, \(X'_1\) is a \(T_1 \delta_1\)–Poincaré complex over \(X_0\); and

(ii) \(p'_1\) is a degree-one normal map with controlled surgery obstruction

\[-p_0^*(\sigma) + \sigma = 0 \in H_n(M; \mathbb{L}).\]

Let \(\xi_1 > 0\) be given. We now apply Theorem 3.1 to produce a \(\xi_1\)–homotopy equivalence by surgeries outside the singular set (note that the surgeries which have to be done are in the manifold part of \(X'_0\)). For this we need a sufficiently small \(\eta_0\)–Poincaré structure on \(X_0\). However, this can be achieved as noted above. This finishes the second step.

We now proceed by induction. What we need for the third step in order to produce \(p_2 : X_2 \to X_1\) is

(i) a degree-one normal map \(g_1 : M_1 \to X_1\) with controlled surgery obstruction \(\sigma \in H_n(X_0; \mathbb{L})\); and

(ii) \(\bar{\sigma} \in H_n(X_1; \mathbb{L})\) with \(p_1^*(\bar{\sigma}) = \sigma\), in the same \(\mathbb{Z}\)–sector as \(\sigma \in H_n(X_0; \mathbb{L})\).

One can get \(g_1 : M_1 \to X_1\) as follows: Consider \(g'_1 : M'_1 \to X'_1\), where

\[M'_1 = B_1 \cup_{\text{Id}} V_1 \cup_{\text{Id}} -V_1 \cup_{\text{Id}} C_1 \to B_1 \cup_{\text{Id}} D_1 \times I \cup_{f_1, \pi} -V_1 \cup_{\text{Id}} C_1 \cong X'_1\]

is induced by \(F_{1,\sigma} : V_1 \to D_1 \times I\) and the identity. The map \(g'_1\) is a degree-one normal map. Then one performs the same surgeries on \(g'_1\) as one has performed on \(p'_1 : X'_1 \to X_0\) to obtain \(X_1\). This produces the desired \(g_1\). For (ii) we note that \(p_1^*\) is a bijective map preserving the \(\mathbb{Z}\)–sectors (since \(p_1^*\) is \(UV^1\)).

So we have obtained the sequence of controlled Poincaré spaces \(p_i : X_i \to X_{i-1}\) and \(p_0 : X_0 \to M\) with degree-one normal maps \(g_i : M_i \to X_i\) and controlled surgery obstructions \(\sigma \in H_n(X_{i-1}; \mathbb{L})\). The properties (iv) and (v) of Section 2 now follow by the thin \(h\)–cobordism theorem and approximation of retraction.

4 Nonresolvability, the DDP property and existence of generalized manifolds

4.1 Nonresolvability

At the beginning of the construction we have \(\sigma \in H_n(M; \mathbb{L})\), where \(M\) is a closed (oriented) \(n\)–manifold with \(n \geq 6\). For each \(m\) we constructed degree-one normal
maps \( g_m: M_m \to X_m \) over \( p_m: X_m \to X_{m-1} \), with controlled surgery obstructions \( \sigma_m \in H_n(X_{m-1}; \mathbb{L}) \), \( p_0(\sigma_1) = \sigma \), \( p_m(\sigma_{m+1}) = \sigma_m \), and all \( \sigma_m \) belong to the same \( \mathbb{Z} \)-sector as \( \sigma \). So we will call all of them \( \sigma \).

We consider the normal map \( g_m: M_m \to X_m \) as a controlled normal map over the identity map \( \text{Id}: X_m \to X_m \), and over \( q_m: X_m \subset W_m \xrightarrow{p} X \) (see Section 2). Since \( \varphi \mid_{\partial W_m} \) is an approximate fibration and \( d(r_i, r_{i-1}) < \xi_i \) and \( \sum_{i=m+1}^{\infty} \xi_i < \varepsilon \), for large \( m \), we can assume that \( q_m \) is \( UV^1(\delta) \) for large \( m \), so \( (q_m)_*: H_n(X_m; \mathbb{L}) \to H_n(X; \mathbb{L}) \) maps \( \sigma \) to \( (q_m)_*(\sigma) = \sigma' \), being in the same \( \mathbb{Z} \)-sector as \( \sigma \). The map \( (q_m)_* \) is a bijective, and we denote \( \sigma' \) by \( \sigma \). In other words, we have a surgery problem

\[
\begin{array}{ccc}
M_m & \xrightarrow{g_m} & X_m \\
\downarrow{q_m} & & \downarrow{X} \\
& & \\
\end{array}
\]

over \( X \), with controlled surgery obstruction \( \sigma \in H_n(X; \mathbb{L}) \). Our goal is to consider the surgery problem

\[
\begin{array}{ccc}
M_m & \xrightarrow{q_m \circ g_m} & X_m \\
\downarrow{\text{Id}} & & \downarrow{X} \\
& & \\
\end{array}
\]

over \( \text{Id}: X \to X \), and prove that \( \sigma \in H_n(X; \mathbb{L}) \) is its controlled surgery obstruction.

Observe that \( q_m \) is a \( \delta \)-homotopy equivalence over \( \text{Id}: X \to X \) if \( m \) is sufficiently large (for a given \( \delta \)).

Let \( \mathcal{N}(X) \cong [X, G/\text{TOP}] \) be the normal cobordism classes of degree-one normal maps of \( X \), and let \( HE_{\delta}(X) \) be the set of \( \delta \)-homotopy equivalences of \( X \) over \( \text{Id}: X \to X \). Our claim will follow from the following lemma.

**Lemma 4.1** Let \( HE_{\delta'}(X) \times \mathcal{N}(X) \xrightarrow{\mu} \mathcal{N}(X) \) be the action map, ie \( \mu(h, f) = h \circ f \). Then for sufficiently small \( \delta' > 0 \), the diagram

\[
\begin{array}{ccc}
HE_{\delta'}(X) \times \mathcal{N}(X) & \xrightarrow{\mu} & \mathcal{N}(X) \\
\downarrow{\text{pr}} & & \downarrow{\Theta} \\
\mathcal{N}(X) & \xrightarrow{\Theta} & H_n(X; \mathbb{L}) \\
\end{array}
\]

commutes.
Proof This follows from Theorem 3.1’ since $HE_{\delta'}(X) \times S_{\epsilon, \delta'}(X, \Id) \to S_{\epsilon, \delta}(X, \Id)$ for sufficiently small $\delta'$ and $\delta''$. \hfill \square

We apply this lemma to the map $HE_{\delta}(X_m, X) \times N(X_m) \to N(X)$, which sends $(h, g)$ to $h \circ g$, where $HE_{\delta}(X_m, X)$ are the $\delta$–homotopy equivalences $X_m \to X$ over $\Id_X$. Let $\psi_m: X \to X_m$ be a controlled inverse of $q_m$. Then $\psi_m$ induces

$\psi_m^*: HE_{\epsilon}(X_m, X) \to HE_{\delta}(X),$

where $\delta$ is some multiple of $\epsilon$. One can then write the following commutative diagram (for sufficiently small $\delta$).

\[
\begin{array}{c}
HE_{\epsilon}(X_m, X) \times H_n(X_m; \mathbb{L}) \\
\uparrow \text{Id} \times \Theta \\
HE_{\epsilon}(X_m, X) \times N(X_m) \xrightarrow{\mu} N(X) \xrightarrow{\Theta} H_n(X; \mathbb{L}) \\
\downarrow (\psi_m)^* \times (q_m)^* \\
HE_{\delta}(X) \times N(X) \xrightarrow{\text{pr}} N(X)
\end{array}
\]

with $HE_{\epsilon}(X_m, X) \times N(X_m) \to H_n(X; \mathbb{L})$ given by $(h, \tau) \mapsto h_*(\tau)$.

It follows from this that for large enough $m$, $q_m \circ g_m: M_m \to X$ has controlled surgery obstruction $\sigma \in H_n(X; \mathbb{L})$. Hence we get non–resolvable generalized manifolds if the $\mathbb{Z}$–sector of $\sigma$ is $\neq 1$.

4.2 The DDP Property

The construction allows one to get the DDP property for $X$ (see [1, Section 8]). Roughly speaking, this can be seen as follows. The first step in the construction is to glue a highly connected cobordism $V$ into a manifold $M$ of dimension $n \geq 6$, in between the regular neighborhood of the 2–skeleton.

The result is a space which has the DDP. The other constructions are surgery on middle–dimensional spheres, which also preserves the DDP. But since we have to take the limit of the $X_m$’s, one must do it more carefully (see [1, Definition 8.1]):

Definition 4.2 Given $\epsilon > 0$ and $\delta > 0$, we say that a space $Y$ has the $(\epsilon, \delta)$–DDP if for each pair of maps $f, g: D^2 \to Y$ there exist maps $\overline{f}, \overline{g}: D^2 \to Y$ such that $d(\overline{f}(D^2), \overline{g}(D^2)) > \delta$, $d(f, \overline{f}) < \epsilon$ and $d(g, \overline{g}) < \epsilon$.

Lemma 4.3 $\{X_m\}$ have the $(\epsilon, \delta)$–DDP for some $\epsilon > \delta > 0$. 

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Proof The manifolds $M^n_m$, for $n \geq 6$, have the $(\varepsilon, \delta)$–DDP for all $\varepsilon$ and $\delta$. In fact, one can choose a sufficiently fine triangulation, such that any $f : D^2 \to M$ can be placed by arbitrary small moves into the $2$–skeleton or into the dual $(n-3)$–skeleton. Then $\delta$ is the distance between these skeleta. The remarks above show that the $X_m$ have the $(\varepsilon, \delta)$–DDP for some $\varepsilon$ and $\delta$.

It can then be shown that $X = \lim \downarrow X_i$ has the $(2\varepsilon, \delta/2)$–DDP (see [1, Proposition 8.4]).

4.3 Special cases

(i) Let $M^n$ and $\sigma \in H_n(M; \mathbb{L})$ be given as above. The first case which can occur is that $\sigma$ goes to zero under the assembly map $A : H_n(M; \mathbb{L}) \to L_n(\pi_1 M)$. Then we can do surgery on the normal maps $F_\sigma : V \to D \times I$, $F_1,\sigma : V_1 \to D_1 \times I$ and so on, to replace them by products. In this case the generalized manifold $X$ is homotopy equivalent to $M$.

(ii) Suppose that $A$ is injective (or is an isomorphism). Then $X$ cannot be homotopy equivalent to any manifold, if the $Z$–sector of $\sigma$ is $\neq 1$. Suppose that $N^n \to X$ were a homotopy equivalence. It determines an element in $[X, G/\text{TOP}]$ which must map to $(1, 0) \in H_n(X; \mathbb{L})$, because its surgery obstruction in $L_n(\pi_1 X)$ is zero and $A$ is injective. This contradicts our assumption that the index of $X$ is not equal to 1. Examples of this type are given by the $n$–torus $M^n = \mathbb{T}^n$.

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