On (co)homology locally connected spaces

Katsuya Eda a, Ummed H. Karimov b, Dušan Repovš c,*

a School of Science and Engineering, Waseda University, Tokyo 169-0072, Japan
b Institute of Mathematics, Academy of Sciences of Tajikistan, Ul. Ainy 299 A, Dushanbe 734063, Tajikistan
c Institute of Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 2964, Ljubljana 1001, Slovenia

Received 8 June 2000; received in revised form 16 January 2001

Abstract

We prove that there exists a cohomology locally connected compact metrizable space which is not homology locally connected. In the category of compact Hausdorff spaces a similar result was proved earlier by G.E. Bredon.

AMS classification: Primary 55N05; 55N10, Secondary 20F12; 55N40; 55N99

Keywords: Čech cohomology; Singular homology; (Co)homology local connectedness; Compact metrizable space; Commutator length

1. Introduction

It is well known that the concept of (co)homology local connectedness plays an important role in the isomorphism theorems of homology and cohomology theories. If a compact metrizable space \( X \) is homology locally connected with respect to the singular homology (abbreviated as \( \text{HLC} \)), then the Borel–Moore, the Čech, the Vietoris, the Steenrod–Sitnikov, and the singular homology groups with integer coefficients of \( X \) are all naturally isomorphic (cf. [2,7,9,10]).

The concepts of (co)homology local connectedness with respect to different homology and cohomology theories are very closely related. For example, the \( \text{HLC} \) property implies the cohomology local connectedness with respect to the Čech cohomology \( (\text{clc}) \) (cf. [2, p. 195]).

* Corresponding author.
E-mail addresses: eda@logic.info.waseda.ac.jp (K. Eda), umed@ac.tajik.net (U.H. Karimov), dusan.repovs@fmf.uni-lj.si (D. Repovš).

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PII: S0166-8641(01)00087-6
Bredon observed that there exists a compact Hausdorff space, which is a clc-space, but not an HLC-space (cf. [2, pp. 130, 131]). Griffiths proved in [4,5] that there exists a compact metric, homology locally connected in Vietoris homology, topological space which is not HLC.

The purpose of the present note is to prove the following theorem which extends the Bredon result [2] and which could be considered as an alternative proof of some Griffiths’ results (cf. [5, p. 477]):

**Theorem 1.1.** There exists a two-dimensional compact metrizable space $X$ such that:

1. $X$ is acyclic in Čech cohomology;
2. $X$ is a clc-space; and
3. $X$ is not an HLC-space.

**2. Preliminaries**

Let $H^*_s(X)$ (respectively $\tilde{H}^*(X)$) denote singular homology (respectively Čech cohomology) groups of a topological space $X$ with integer coefficients. A finite-dimensional space $X$ is said to be homology (respectively cohomology) locally connected, and called an HLC-space (respectively, clc-space), if for every point $x \in X$ and every neighborhood $U_x \subset X$ of $x$ there exists a neighborhood $V_x \subset U_x$ of $x$ such that the inclusion-induced homomorphism $H^*_s(V_x, \{x\}) \to H^*_s(U_x, \{x\})$ (respectively $\tilde{H}^*(U_x, \{x\}) \to \tilde{H}^*(V_x, \{x\})$) is zero.

Let $g \in G$ be an arbitrary element of a group $G$. By the commutator length $\text{cl}(g)$ of $g$ we shall denote the minimal number of commutators of the group $G$ whose product is equal to $g$, i.e.,

$$\text{cl}(g) = \min \{n \in \mathbb{N} \mid g = [g_1, g_2] \circ \cdots \circ [g_{2n-1}, g_{2n}], \text{ for some } g_i \in G\}.$$ 

If such a number does not exist then we set $\text{cl}(g) = \infty$ (cf. [3, Definition 4.15]).

Since clearly, for every $g_i, h \in G$, one has that $[g_1, g_2]^{-1} = [g_2, g_1]$ and

$$h \circ [g_1, g_2] \circ h^{-1} = [h \circ g_1 \circ h^{-1}, h \circ g_2 \circ h^{-1}],$$

it follows that

$$\text{cl}(g) = \infty \text{ if and only if } g \notin G',$$ \hspace{1cm} (1)

where $G' = [G, G]$ is the commutator subgroup of $G$.

If $\varphi : G_1 \to G_2$ is any homomorphism between groups $G_1$ and $G_2$, then for every $g \in G_1$ clearly,

$$\text{cl}(\varphi(g)) \leq \text{cl}(g).$$ \hspace{1cm} (2)

For every path connected space $X$, its fundamental group $\pi_1(X)$ does not depend on the choice of the base point and

$$H^*_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)].$$ \hspace{1cm} (3)
The compact bouquet of topological spaces $X_i$ rel points $x_i \in X_i$, $i \in \mathbb{N}$, is defined as the quotient space of the topological sum $\bigsqcup_{i=1}^{\infty} X_i$ by the subset $\bigsqcup_{i=1}^{\infty} \{x_i\}$, equipped with the strong topology, i.e., a set $U \subset \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty} \{x_i\}$ is open if and only if:

1. for every $i \in \mathbb{N}$, the set $\Pi^{-1}(U) \cap X_i$ is open in $X_i$, where

$$\Pi : \bigsqcup_{i=1}^{\infty} X_i \to \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty} \{x_i\}$$

is the canonical projection onto the quotient space; and

2. if $U$ contains the point $\bar{x} \in \bigsqcup_{i=1}^{\infty} X_i / \bigsqcup_{i=1}^{\infty} \{x_i\}$ which corresponds to the points $x_i$, then there exists an index $n_0 \in \mathbb{N}$ such that $\Pi(X_j) \subset U$, for all $j \geq n_0$.

The point $\bar{x}$ is called the base point of the bouquet $X$. We shall denote the compact bouquet of spaces $X_i$ rel the points $x_i \in X_i$, $i \in \mathbb{N}$, by $(X, \bar{x}) = \bigsqcup_{i=1}^{\infty} (X_i, x_i)$.

3. Proof of Theorem 1.1

Let for every integer $i \in \mathbb{N}$, $P_i$ be any finite acyclic 2-dimensional polyhedron with a nontrivial fundamental group—for example, take the 2-polyhedron constructed by any of the presentations (cf., e.g., [1]):

$$\{a, b | b^{-2} \circ a \circ b \circ a, b^{-3k} \circ a^{6k-1}\}$$

or

$$\{a_1, \ldots, a_r | a_1 \circ a_2 \circ a_1^{-1} \circ a_2^{-2}, a_2 \circ a_3 \circ a_2^{-1} \circ a_3^{-2}, \ldots, a_r \circ a_1 \circ a_r^{-1} \circ a_1^{-2}\},$$

where $r > 3$ (cf. [6]).

Let furthermore $X_i$ denote the bouquet $P_i \lor P_i$. For every $i \in \mathbb{N}$, choose a point $x_i \in X_i$ and let $X$ be the compact bouquet of $X_i$’s rel the points $\{x_i\}$, i.e.,

$$(X, \bar{x}) = \bigsqcup_{i=1}^{\infty} (X_i, x_i).$$

By the continuity property of Čech cohomology we can conclude that $\check{H}^*(X, \bar{x}) = 0$. Indeed, $X$ is the inverse limit of the following spectrum:

$$\left\{ \bigsqcup_{i=1}^{i=n} X_i \left\downarrow\right|_{i=1}^{i=n+1} X_i \right\}_{n \in \mathbb{N}}.$$

Since the polyhedra $X_i$ are acyclic, $X$ is acyclic with respect to Čech cohomology, i.e.,

$$\check{H}^*(X, \bar{x}) = 0.$$

Since every point of $X \setminus \{\bar{x}\}$ has a closed polyhedral neighborhood and the point $\bar{x}$ has arbitrary small closed neighborhoods in $X$ which are all homotopy equivalent to $X$, the space $X$ is a clc-space.

Since clearly, for every $i \in \mathbb{N}$,

$$\pi_1(X_i) = \pi_1(P_i) \ast \pi_1(P_i),$$

(4)
there exists, by [3, Lemma 4.17], for every $i \in \mathbb{N}$, an element $q_i \in \pi_1(X_i)$ such that:

$$i < \text{cl}(q_i) < \infty,$$

for every $i \in \mathbb{N}$.  \(5\)

Consider now the unit circle $S^1 \subset \mathbb{C}$ and its closed subset

$$A = \{e^{i\varphi} \in S^1 \mid \varphi = 2\pi/k, \ k \in \mathbb{N}\}.$$

The quotient space $S^1/A$ is homeomorphic to the Hawaiian earring $H$, i.e., to the compact bouquet of a countable number of circles $\{S^1_i\}_{i \in \mathbb{N}}$. Let $p : S^1 \to H$ be the canonical projection.

For every $i \in \mathbb{N}$, let $f_i : S^1_i \to X_i$ be a representative of $q_i$, which maps the base point to the base point. Next, let $f : H \to X$ be a continuous mapping such that for every $i \in \mathbb{N}$, its restriction onto $S^1_i$ is $f_i$. Finally, let $g = f \circ p : S^1 \to X$.

Suppose that $H^s_1(X) = 0$. Then by (1) and (3),

$$\text{cl}([g]) < n_0 < \infty,$$

for some $n_0 \in \mathbb{N}$.

Let $p_{n_0} : X \to X_{n_0}$ be the canonical projection and $p_{n_0}^* : \pi_1(X) \to \pi_1(X_{n_0})$ the induced homomorphism of fundamental groups. Then it follows by (2) that

$$\text{cl}(p_{n_0}[g]) = \text{cl}([g]) < n_0.$$

However, by (5) we can conclude that

$$n_0 < \text{cl}(q_{n_0}) \text{ and } q_{n_0} = p_{n_0}^*[g].$$

This is a contradiction. Therefore $H^s_1(X) \neq 0$. It now follows by the Universal Coefficient Theorem, that $H^s_1(X, \bar{x}) \neq 0$. Since by (4), $\tilde{H}^s_1(X, \bar{x}) = 0$, it follows that the compactum $X$ cannot be an HLC-space (for HLC-spaces, the Čech cohomology groups are naturally isomorphic to the singular cohomology groups—cf. [2]). \(\square\)

**Remark 3.1.** It can be shown [3, Theorem 4.14] that the group $H^s_1(X)$ of the space $X$ constructed in the proof above contains a torsion-free divisible group of the cardinality of the continuum. On the other hand, the group $H^s_1(X)$ is trivial. In order to verify this, it suffices to show that every homomorphism from $\pi_1(X)$ to $\mathbb{Z}$ is trivial.

Indeed, by [3, Theorem A.1], $\pi_1(X)$ is the free $\sigma$-product of the groups $\pi_1(P_i \lor P_i)$. Next, by [3, Proposition 3.5 and Corollary 3.7], any homomorphism from $\pi_1(X)$ to $\mathbb{Z}$ factors through the free product of finitely many groups $\pi_1(P_i \lor P_i)$. Hence the assertion follows.

**Remark 3.2.** Since $H^s_1(X) \neq 0$, it follows by the Mayer–Vietoris exact sequence that none of the $k$th suspensions $\Sigma^k(X)$ is acyclic and hence none of them is contractible. On the other hand, it follows by the Seifert–van Kampen and the Hurewicz Theorems that all suspensions

$$\Sigma^k\left(\bigvee_{i=1}^{i=n} X_i\right)$$

are contractible spaces (cf. [8]).
**Question 3.3.** Let $G$ be one of the following two types of groups:

\[ \{a, b \mid b^{-2} \circ a \circ b \circ a, b^{-3k} \circ a^{6k-1} \} \]

or

\[ \{a_1, \ldots, a_r \mid a_1 \circ a_2 \circ a_1^{-1} \circ a_2^{-2}, a_2 \circ a_3 \circ a_2^{-1} \circ a_3^{-2}, \ldots, a_r \circ a_1 \circ a_r^{-1} \circ a_1^{-2} \} \],

where $r > 3$ (cf. [6]). Does it then follow that: $\text{sup}\{\text{cl}(g) : g \in G\} = \infty$?

**Acknowledgements**

We wish to acknowledge remarks from the referee. The second author wishes to express his gratitude to Narzullo C. Mirzobaev for everything he has learned from him. The third author acknowledges the support of the Ministry for Science and Technology of the Republic of Slovenia.

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